WAVELETS: AN INTRODUCTION

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Abstract

The essential elements of the theory of wavelets are presented in a very simple manner. After an intuitive motivation (1) the form of the wavelet series is presented (2.1); the concepts of the multiresolution analysis and the two scale difference equations are introduced as tools for the construction of wavelet bases for functions of one (2.3) and of several (2.3) variables. Two examples of wavelets are presented (2.4) and the result of the wavelet analysis of a synthetic signal is illustrated by some plots (3). Suggestions for finding resources on the network and for further reading are presented (4).

1. WHY WAVELETS ?

Is there any good reason why an applied scientist, like a photogrammetrist, would start studying a new, modern and advanced mathematical topic ? The aim of this paper is just to show that the topic of wavelets, although deeply founded on advanced mathematics, can be presented in its essential lines in an (almost) elementary way and can provide useful tools for the treatment of several practical problems. I will introduce the topic by means of a comparison of two well known techniques for the approximation of a function: finite elements and Fourier series. For the first we have:

$$f(x) = \sum a_n g(x-n) \qquad (1)$$

where $g(x)$ is e.g. piecewise linear, supported in $-1..+1$, and continuous; the second is:

$$\begin{aligned} f(x) &= \\ &= \sum (a_m \cos(2\pi m x) + b_m \sin(2\pi m x)) = \\ &= \sum c_m e^{i2\pi m x}. \qquad (2) \end{aligned}$$

We can now pay attention to some relevant differences between (1) and (2), specially differences in the building-blocs of the two expressions, i.e. the spline functions and the harmonic functions. The functions $g(x-n)$ are compactly supported and they are all *translated replica of the same function*; the index n represent a translation.On the other hand the harmonic functions $\sin(2\pi m x)$ and $\cos(2\pi m x)$ are globally supported (and do not exhibit any localisation anywhere) and they are *dilated replica of the same two functions*; the index m represent dilation.

A further insight in the comparison can be done looking at the Fourier transform of the base functions: the transform of $g(x-n)$ is a smooth globally supported function that only varies its phase when n is changed. The transform of the complex exponential $e^{i2\pi m x}$ is a (generalized) function supported in a single point for which the parameter m can be interpreted as a dilation or as a translation.

The advantage of the representation (1) are just related to the localisation property of the base-function that allows a tailored representation of compactly supported signals, transient signals and, with some modification, irregularly sampled signals. On the other hand the expression (2) allows an immediate computation of operators such as derivatives of any order and many other simply acting on the coefficients; furthermore the coefficients themselves give information about the periodic components of the signal. On the contrary the approximation of a function with a localised feature (e.g. a signal with a local discontinuity or a signal with compact support, or finite duration) by means of (2) is very expensive, because it requires a lot of terms.

If we wont to retain the advantages of finite elements approximation and that of Fourier series we need a set of base function with good localisation property both in the natural and in the frequency domain. The possibility of finding such functions is limited by some mathematical properties of the Fourier transform, but some compromise solutions are nowadays available, and wavelets are the more flexible of them.

2. ELEMENTS OF WAVELETS THEORY

2.1 An heuristic introduction to wavelet series

Following our intuitive approach (and with no mind for mathematical rigor) we try to define the wavelet transform as an analysis and synthesis tool that uses a set of base-functions characterised by two indices, one related to translation (as n in (1)) and the other to dilation (as m in (2)), we can even use base-functions that are dilated and translated version of the same mother function. This mean that we wont a representation of $f(x)$ in the form

$$f(x) = \sum_{m,n} f_{m,n}\psi_{m,n}(x) \tag{3}$$

where the base functions are:

$$\psi_{m,n}(x) = 2^{-m/2}\psi(2^{-m}x - n). \tag{4}$$

In this case we have the classical choice of dyadic dilation and integer translation, but different choices are possible. It is also possible to represent $f(x)$ as an integral transform using a dilation and a translation parameter that vary with continuity, but this will not be treated here. It must be noted that we have written (3) and (4) just on the basis of heuristic arguments (i.e. looking for base functions with a dilation index and a translation index), but untill now we have no any idea if a function $\psi(x)$ exist such that the series (3) exhibit some reasonable behaviour. There are several techniques for the construction of such functions, one of this techniques is the so called multiresolution analysis, that will be treated here after.

2.2 Multiresolution Analysis and Wavelet Bases of $L^2(R)$

This paragraph uses the language of basic functional analysis; the reader is awarded that the entities introduced here after have a quite technical definition, but the main concepts are quite simple and can be explained in an intuitive manner as it will be done in the end.

A multiresolution analysis is a sequence of subspaces of $L^2(R)$ with the properties:

$$\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots \tag{5}$$

$$\overline{\bigcup_m V_m} = L^2(R) \tag{6}$$

$$\bigcap_m V_m = \{0\} \tag{7}$$

$$f(x) \in V_m \Longleftrightarrow f(2^m x) \in V_0 \tag{8}$$

$$f(x) \in V_0 \Longleftrightarrow f(x-n) \in V_0 \tag{9}$$

furthermore we require the existence of a function $\phi(x)$ such that the sequence: $\phi_{0,n}(x) = \phi(x-n)$ is an orthonormal base of V_0 and the sequence (indexed by n):

$$\phi_{m,n}(x) = 2^{-m/2}\phi(2^{-m}x - n) \tag{10}$$

is a n orthonormal base of V_m. (In practice the function $\phi(x)$ and the space V_0 are often defined simultaneously).

The relations

$$V_{m-1} = V_m \oplus W_m \tag{11}$$

and

$$W_m \perp V_m \tag{12}$$

define the orthogonal complement W_m of V_m in V_{m-1}, that is the space of functions that belongs to V_{m-1} but not to V_m. As a consequence of all these definitions we have:

$$L^2(R) = \bigoplus_m W_m. \tag{13}$$

The subspaces $\{W_m\}$ have the same scaling property of $\{V_m\}$:

$$f(x) \in W_m \Longleftrightarrow f(2^m x) \in W_0 \tag{14}$$

and it is possible to construct a function $\psi(x)$ such that its translated versions:

$$\psi_{0,n}(x) = \psi(2^{-m}x - n) \tag{15}$$

are an orthonormal base of W_0 while its dilated and translated versions:

$$\psi_{m,n}(x) = 2^{-m/2}\psi(2^{-m}x - n) \tag{16}$$

are an orthonormal base of W_m. As a consequence of (13), the functions $\psi_{m,n}(x)$ are all together a wavelet orthonormal base of $L^2(R)$.

Now we can look at the *intuitive meaning* of all the above definitions. With a proper choice of V_0 and of $\phi(x)$, the spaces V_i contains smoother and smoother functions when the index i increases. We can consider the projections of a function $f(x)$ into V_i (i.e. the best approximation of the considered function included in the various subspaces); this projections are smoother and smoother when the index i increases and, on the contrary, are more and more detailed when the index decreases. The projection of $f(x)$ into W_i represents the details that are in V_{i-1} but not in V_i.

It has already been stated that the functions $\psi_{m,n}(x)$ form an orthonormal base of $L^2(R)$; in practice this mean that it is possible to represent any square integrable function in the form of a series like (3) and that the coefficients can be expressed as

$$f_{nm} = \langle f(x), \psi_{nm}(x) \rangle \tag{17}$$

where the notation $\langle .,. \rangle$ denotes the scalar product of two functions, i.e. $\langle f(x), g(x) \rangle = \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{d}x$

The existence and the properties of the functions $\phi(x)$, the scaling function or *the father of wavelets*, and $\psi(x)$, *the mother of wavelets*, can be studied by means of the so called two-scale difference equation or *dilation equation* which is also important for the practical computation of the wavelet analysis and synthesis.

The dilation equation is:

$$\phi(x) = \sqrt{2}\sum_n h_n \phi(2x - n) \tag{18}$$

where:

$$h_n = \langle \phi, \phi_{-1,n} \rangle \tag{19}$$

and

$$\sum_n h_n^2 = 1. \tag{20}$$

In fact we have: $\phi \in V_0 \subset V_{-1}$, and $\phi_{-1,n} = \sqrt{2}\phi(2x - n)$ are an orthonormal base of V_{-1}; therefore ϕ can be expanded in the form (18). The expression for ψ in term of the same base of V_{-1} is:

$$\psi(x) = \sqrt{2} \sum_n (-1)^{n-1} h_{-n-1} \phi(2x - n) \tag{21}$$

which links together the functions ψ and ϕ. (The expression (21) can be obtained from the property of W_0 with some manipulations). Two important points must be noted here:
- the choice of the coefficients h_n completely characterises both $\phi(x)$ and $\psi(x)$ and,
- the computation of the wavelet analysis and synthesis can be performed using just the coefficient h_n without the explicit evaluation of the functions $\phi(x)$ and $\psi(x)$.

2.3 Multiresolution Analysis and Wavelet Base of $L^2(R^N)$

There are several techniques for the construction of bases of wavelet in more than one dimension (the two dimensional case will be treated here). One of these techniques is based on the tensor product of the nested spaces V_n defined in the previous paragraph. This construction is specially useful because it produces a multidimensional base of wavelets where each element of the base has the same scale in all the different directions. The following relation defines the sequence of nested subspaces $\mathbf{V}_m$ of $L^2(R^2)$:

$$\mathbf{V}_m = \overset{x}{V}_m \otimes \overset{y}{V}_m \tag{22}$$

and the following two relations define their complements $\mathbf{W}_m$:

$$\mathbf{V}_{m-1} = \mathbf{V}_m \oplus \mathbf{W}_m \tag{23}$$

$$\mathbf{W}_m \perp \mathbf{V}_m. \tag{24}$$

($\overset{x}{V}_m$ and $\overset{y}{V}_m$ are the nested spaces of functions of x and y, respectively.) From the above definitions and with simple manipulations it results:

$$\mathbf{W}_m = (\overset{x}{V}_m \otimes \overset{y}{W}_m) \oplus \oplus (\overset{x}{W}_m \otimes \overset{y}{W}_m) \oplus \oplus (\overset{x}{W}_m \otimes \overset{y}{V}_m) \tag{25}$$

therefore the space $\mathbf{W}_m$ is defined in term of combinations of spaces of functions of 1 variable and the set

$$\{\phi_{m,n_x}(x)\psi_{m,n_y}(y); \; \psi_{m,n_x}(x)\psi_{m,n_y}(y); \; \psi_{m,n_x}(x)\phi_{m,n_y}(y)\} \tag{26}$$

is (for fixed m) an orthonormal base of $\mathbf{W}_m$ and it is a base of $L^2(R^2)$ if all the indices run from $-\infty$ to $+\infty$. Each element of the set (26) is characterized by a single scale parameter m and by two location parameters n_x and n_y.

2.4 Examples of Wavelets

There are several possibilities for the definition of a wavelet base and the related multireslution analysis. The simplest example of wavelet base is the Haar base, whell known far before the modern development of the theory of multiresolution analysis. The space V_0 is defined here as the space of piecewise constant functions with discontinuities only localised on the integers. The function ϕ is:

$$\phi(x) = \begin{cases} 1, & \text{for } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \tag{27}$$

and the function ψ is:

$$\psi(x) = \begin{cases} 1, & \text{for } 0 < x = \frac{1}{2} \\ -1, & \text{for } \frac{1}{2} < x < 1 \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

For the Haar function we have only two non-zero coefficients in the dilation equations:

$$h_0 = \frac{1}{\sqrt{2}} \text{ and } h_1 = \frac{1}{\sqrt{2}}$$

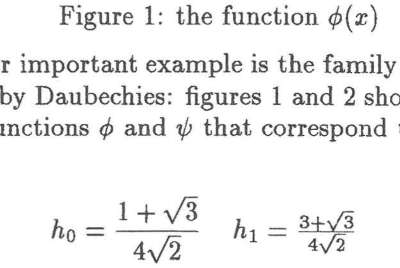

Figure 1: the function $\phi(x)$

An other important example is the family of wavelets defined by Daubechies: figures 1 and 2 show the plots of the functions ϕ and ψ that correspond to the coefficients:

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}} \quad h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$$

$$h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}} \quad h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

The main feature of this family of wavetes is that they are compactly supported and fully orthonormal. With more non null h coefficients we have smoother functions but also a wider support.

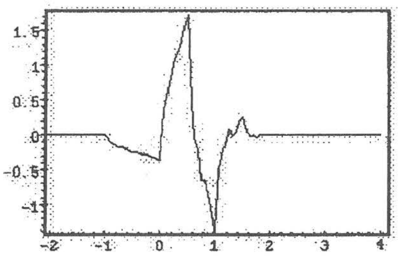

Figure 2: the function $\psi(x)$

3. THE ANALYSIS OF A SIMULATED DATA SET

This example is just illustrated by some figures and some comments. The synthetic dada set plotted in figure 3 exhibits a very different behaviour in the first and second half of the covered interval.

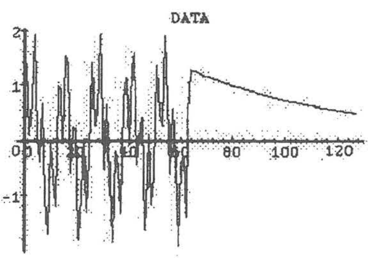

Figure 3: A synthetic signal

It can be seen by direct computation that the Fourier transform is totally blind with respect to this particular aspect of the data set. On the other hand the wavelet analysis *respects* the different character of the data in the different parts of the support. At each step of the wavelet analysis the function is treated by an *averaging* filter and by a *differentiating* filter. The coefficients obtained by the differentiating filter are stored step by step and represents the main result of the analysis, the coefficients obtained by the averaging filter are stored only temporarily and are used as input data for the following step. Teh figure 4 represents the result of the first step of the analysis: the "average " coefficients are plotted by circles "o", the "difference" coefficients are plotted by crosses "+" and are very small where the function is smooth; this is due to the localisation of the base functions and allows an efficient compression of the signal.

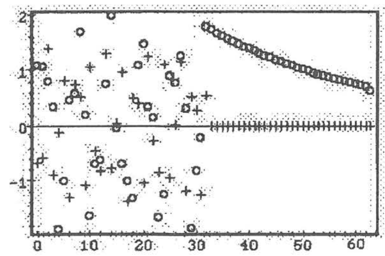

Figure 4: A single step wavelet analysis

4. RESOURCES AND BIBLIOGRAPHY

4.1 Resources on the net

There are a lot of useful resources freely available on the net. I refer here after just a couple of sites that are worth to be visited by anybody interested in the topic of wavelets. (The citation is already in HTML format.)

<A HREF="http://www.wavelet.org/"> Wavelet Digest </A>

<A HREF="http://www.tsc.uvigo.es/~wavelets/uvi_wave.html"> Univ. of VIGO (Spain) Wavelet page </A> contains documents about wavelets and a downlodable package for wavelets computations inMatLab

4.2 Bibiography

This is a very essential bibliography about wavelets. Only textbooks are mentioned.

Chui C.K. *An introduction to wavelets* Academic Press (1992) ISBN 0-12-174584-8

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Strang G., Nguyen T. *Wavelets and filter banks* Wallesley-Cambridge Press (1996) ISBN 0-9614088-7-1