THE COPLANARITY CONDITION FOR THE ORIENTATION IN SURVEYING

ISPRS WG VI/3

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KEY WORDS: **Surveying, Close-Range Photogrammetry, Education**

ABSTRACT

The coplanarity condition in photogrammetry is used for the determination of the relative position of one bundle with respect to the other one. In a similar manner, in surveying, by imposing the coplanarity condition, in the intersection, the relative orientation of one theodolite station can be determined, with respect to the other one without the link of distances neither of directions, without the knowledge of the station co-ordinates. A minimum of three points must be observed. The final ground co-ordinates are computed in a sort of absolute orientation, where at lest two control points in the final reference system should be supplied. These two points can be surveyed by means of a simple metric tape. Approximated values of the orientation angles should be given, with an approximation of some ten degrees. Approximated values for linear orientation parameters can be set to zero. The larger component of the orientation errors is corrected in the absolute orientation The described procedure is mainly helpful in short-range local triangulations. The coplanarity algorithm is described. This approach overcomes the indetermination where the observed points and the station points lie on the same circle. Some examples are shown and described. The achieved accuracy is comparable to the classical intersection. Compared with photogrammetry, the algorithm is more efficient and less unstable, because it has less unknown orientation parameters. Finally the procedure is helpful for education to show the students the differences from photogrammetry to surveying. The triple intersection increases very much the redundancy.

1. INTRODUCTION

The equation of coplanarity in photogrammetry is used for the determination of the five parameters of the relative orientation i.e. for the determination of the relative position of the two projective bundles formed by the photogrammetric model. The co-ordinates of the intersected points are given by the intersection of the corresponding projective straight lines. The final co-ordinates in the absolute reference system are got with a roto-translation with scale variation in the space. A similar procedure can be employed in surveying also, in the case of the intersection from two or more theodolite stations. The advantage of such an approach consists in the possibility to avoid to measure the directions and the distance linking the two stations. The coplanarity finds the relative position of the stations minimising, in the sense of the least squares, the distance between the corresponding intersecting straight lines. Such a distance is in practice the difference between the elevations of the intersected point, elevations coming from the two stations, analogue to the transversal parallax in photogrammetry.

Let S1 and S2 be the two theodolite stations from where an unknown point P (X,Y,Z) is intersected. Similarly to photogrammetry, one can think that the theodolite from S_1 and S_2 intersects in P' and in P'' two fictitious vertical

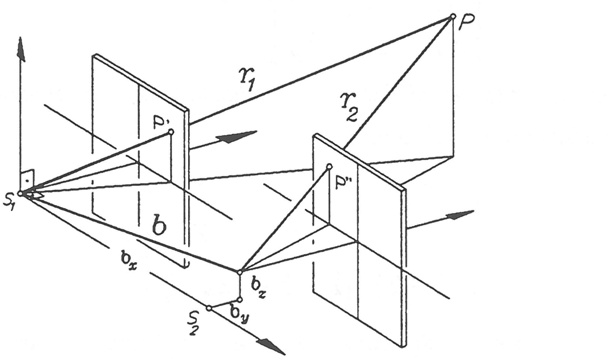


*Fig. 1 - Intersection - The three coplanar vectors **b**, r_1 and r_2..*

image plates, parallel to each other, with projection centres in S_1 and S_2 and principal distance ***c*** .

The measures from S_1 and S_2 are the horizontal directions l' and l'' and the vertical angles φ' and φ'' .

In photogrammetry the two projective bundles are defined by 6 + 6 =12 orientation parameters. In this case, for the particular nature of the surveying measures, the orientation parameters are only 4+4 =8 :

- the 3 + 3 co-ordinates of points of station X_{S1}, Y_{S1} Z_{S1} and X_{S2}, Y_{S2} Z_{S2}
- the 2 unknown horizontal orientation angles ϑ' and ϑ'' (bearings).

The determination of the 8 unknowns of orientation can be solved in two steps:

1. **relative orientation**: solution of the 3 parameters of the relative orientation of a station with respect to the other one;
2. **absolute orientation:** a plane roto-translation with scale variation in the plane (4 parameters) and a translation in elevation (fifth parameter).

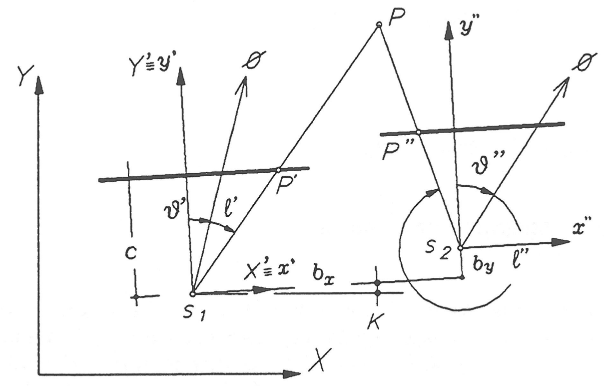


Fig. 2 - Intersection - The absolute reference system and the relative systems

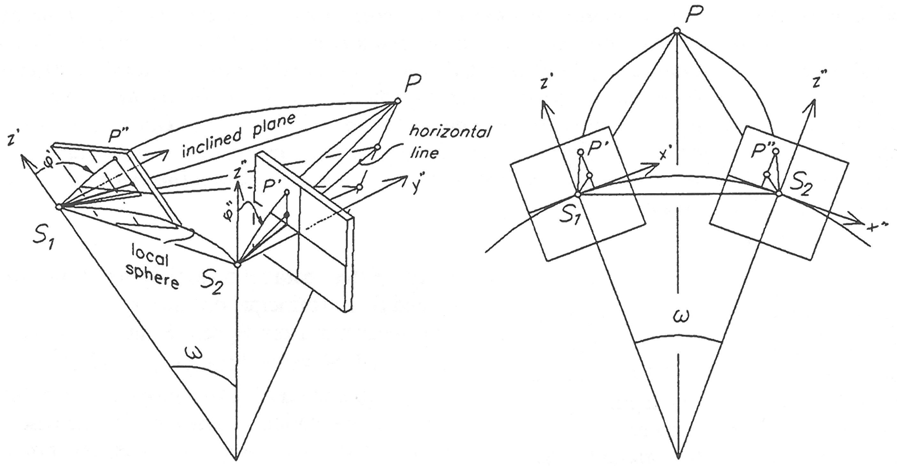


Fig. 3 - Intersection with the sphericity and refraction effects

The four points S_1 , S_2 , P' and P" are coplanar neglecting the effect of sphericity and refraction.

$$\begin{Vmatrix} X_{S1} & Y_{S1} & Z_{S1} & 1 \\ X_{S2} & Y_{S2} & Z_{S2} & 1 \\ X_{P'} & Y_{P'} & Z_{P'} & 1 \\ X_{P''} & Y_{P''} & Z_{P''} & 1 \end{Vmatrix} = 0 \qquad (1)$$

where $X_{P'}$, $Y_{P'}$, $Z_{P'}$ and $X_{P''}$, $Y_{P''}$, $Z_{P''}$ are the co-ordinates of the image points P' and P". Let's put the origin of the reference system in the first station point S_1: the second station point S_2 has co-ordinates equal to the components (b_x, b_y, b_z) of the base b[1] (figs. 1 and 2). The coplanarity condition of the points S_1,S_2,P', P" becomes:

The coplanarity condition of the three vectors ***b r1*** and ***r2*** (fig.1) is

$$\begin{Vmatrix} 0 & 0 & 0 & 1 \\ b_x & b_y & b_z & 1 \\ X_{P'} & Y_{P'} & Z_{P'} & 1 \\ X_{P''} & Y_{P''} & Z_{P''} & 1 \end{Vmatrix} = 0 \qquad (2)$$

or for known properties of the determinants:

$$\begin{Vmatrix} b_x & b_y & b_z \\ X_{P'} & Y_{P'} & Z_{P'} \\ X_{P''} & Y_{P''} & Z_{P''} \end{Vmatrix} = 0 \qquad (3)$$

Again for the proprieties of the determinants one can subtract the first row to the last one.

$$\begin{Vmatrix} b_x & b_y & b_z \\ X_P & Y_P & Z_P \\ (X_{P''}-b_x) & (Y_{P''}-b_y) & (Z_{P''}-b_z) \end{Vmatrix} = 0 \qquad (4)$$

Let's suppose to have two new reference systems (x', y', z'), (x'', y'', z'') with origin in the two station

[1]The adopted reference systems in the present note are (fig.1):

- (X, Y, Z) absolute co-ordinates
- (X', Y', Z') co-ordinates relative (with origin in S_1)
- (x', y', z'), (x'', y'', z'') systems parallel to (X', Y', Z').,with origin respectively in S_1 and S_2.

points and parallel to the relative reference system (X',Y',Z') (fig.2), then the coplanarity condition becomes:

$$\begin{Vmatrix} b_x & b_y & b_z \\ x' & y' & z' \\ x'' & y'' & z'' \end{Vmatrix} = 0 \qquad (5)$$

that can be written:

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} \cdot \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix} \cdot \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = 0 \qquad (6)$$

that developed gives:

$$-y'b_z x'' + z'b_y x'' + x'b_z y'' - z'b_x y'' - x'b_y z'' + y'b_x z'' = 0 \qquad (7)$$

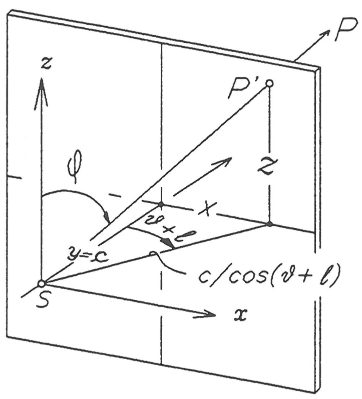


Fig. 4 - The fictitious "plate" co-ordinates x, y, z and the observed quantities l, φ and the unknown orientation ϑ.

Each couple of measures l *and* φ corresponds to a couple of fictitious plate co-ordinates x, z. All the "image plate" points have equal ordinate $y=c$. Then it holds (fig.4)

$$x' = c \cdot \tan(\vartheta'_0 + l'_P) \quad x'' = c \cdot \tan(\vartheta''_0 + l''_P) \qquad (8)$$

$$y' = y'' = c \qquad (9)$$

and the elevations

$$Z' = \frac{c \cdot cotg\varphi'}{\cos(\vartheta' + l')}$$

$$Z'' = \frac{c \cdot cotg\varphi''}{\cos(\vartheta'' + l'')} \qquad (10)$$

ignoring the sphericity and refraction effects.

$$f = -b_z \cdot tg(\vartheta'' + l'') + \frac{cotg\varphi'}{cos(\vartheta' + l')} b_y \cdot tan(\vartheta'' + l'') + b_z \cdot tan(\vartheta' + l')$$
$$- \frac{cotg\varphi'}{cos(\vartheta' + l')} b_x - \frac{cotg\varphi''}{cos(\vartheta'' + l'')} b_y \cdot tan(\vartheta' + l')$$
$$+ \frac{cotg\varphi''}{cos(\vartheta'' + l'')} b_x = 0 \qquad (11)$$

We can set the fictitious principal distance C equal to the unit and divide by b_x eq.(2), homogeneous in the unknown parameters, setting $\beta_y = b_y / b_x$ and $\beta_z = b_z / b_x$; eq. (11) becomes:

$$f = -\beta_z \cdot tg(\vartheta'' + l'') + \frac{cotg\,\varphi'}{cos(\vartheta' + l')} \beta_y \cdot tan(\vartheta'' + l'') + \beta_z \cdot tan(\vartheta' + l')$$
$$- \frac{cotg\,\varphi'}{cos(\vartheta' + l')} - \frac{cotg\,\varphi''}{cos(\vartheta'' + l'')} \beta_y \cdot tan(\vartheta' + l') + \frac{cotg\,\varphi''}{cos(\vartheta'' + l'')} = 0 \qquad (12)$$

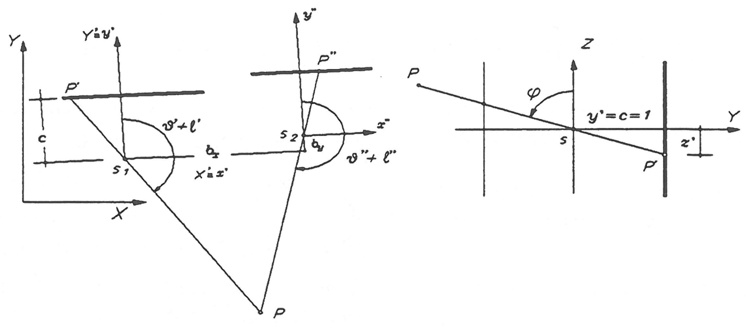


Fig. 5 - The observed point P is in the opposite side that in fig. 1

Differently from photogrammetry, where all the object points are on the same side of the photogram and projection centre, in this case the object point *P* can be placed anywhere around the station point. The (12) is still valid; in fact in eq.(10) $\cos(\vartheta + l)$ becomes negative and also the co-ordinate plate Z is negative (fig.5).

2. DETERMINATION OF THE PARAMETERS OF THE RELATIVE ORIENTATION

The four parameters of the relative orientation are ϑ', ϑ'', β_y, e β_z. Only three of them are independent.

The equation of coplanarity (11) is not linear in the unknowns. Its Taylor's expansion is, indicated with f_0 the initial approximated value of the function f:

$$f_0 + \Delta f = f_0 + \frac{\partial f}{\partial \vartheta'} d\vartheta' + \frac{\partial f}{\partial \vartheta''} d\vartheta'' + \frac{\partial f}{\partial b_x} db_x + \frac{\partial f}{\partial b_y} db_y + \frac{\partial f}{\partial b_z} db_z = 0 \tag{13}$$

The partial derivatives are:

$$\begin{aligned}\frac{\partial f}{\partial \vartheta'} &= \frac{\cot g\varphi' \cdot sin(\vartheta'+l')}{\cos^2(\vartheta'+l')}\beta_y \cdot tg(\vartheta''+l'') + \frac{\beta_z}{cos^2(\vartheta'+l')} + \\ &- \frac{sin(\vartheta'+l')}{cos^2(\vartheta'+l')\cdot tan\varphi'} - \frac{\beta_y \cdot \cot g\varphi''}{\cos(\vartheta''+l'')\cos^2(\vartheta'+l')} = c' \cdot \left[-t'b_x + (x''t'-z'')b_y + b_z\right]\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial f}{\partial \vartheta''} &= \frac{-\beta_z}{cos^2(\vartheta''+l'')} + \frac{\beta_y \cot g\varphi'}{\cos(\vartheta'+l')\cos^2(\vartheta''+l'')} \\ &- \frac{tg(\vartheta'+l')\beta_y \cdot sin(\vartheta''+l'')}{\cos^2(\vartheta''+l'')\cdot tg\varphi''} + \frac{sin(\vartheta''+l'')}{cos^2(\vartheta''+l'')\cdot tan\varphi''} = -c'' \cdot \left[-t''b_x + (x't''-z'')b_y + b_z\right]\end{aligned} \tag{15}$$

$$\frac{\partial f}{\partial b_x} = -\frac{cotg\varphi'}{cos(\vartheta'+l')} + \frac{cotg\varphi''}{cos(\vartheta''+l'')} = z''-z' \tag{16}$$

$$\frac{\partial f}{\partial b_y} = \frac{\cot g\varphi'}{\cos(\vartheta'+l')} tan(\vartheta''+l'') - tan(\vartheta'+l') = x''z' - x'z'' \tag{17}$$

$$\frac{\partial f}{\partial b_z} = -tan(\vartheta''+l'') + tan(\vartheta'+l'') = x' - x'' \tag{18}$$

having set:

$$a' = \vartheta'+l' \qquad a'' = \vartheta''+l'' \qquad z' = \frac{\cot g\varphi'}{\cos a'} \qquad z'' = \frac{\cot g\varphi''}{\cos a''}$$

$$x' = \tan a' \qquad x'' = \tan a'' \qquad c' = 1/\cos^2 a' \qquad c'' = 1/\cos^2 a''$$

$$t' = sina_1/\tan\varphi' \qquad t'' = sina''/\tan\varphi'' \; . \tag{19}$$

Equ.(11) is then:

$$f = (z''-z')\cdot b_x + (x''z' - x'z'')\cdot b_y + (x'-x'')\cdot b_z = 0 \tag{20}$$

Every point P supplies one observation equation. At least three points (not aligned) are needed to compute the orientation parameters.
It could seem that there could be as many types of relative orientation as many are the combinations of 4 elements 3 to 3 say $\binom{4}{3} = 4$, but in order to distinguish the vertical measures from the planimetric ones, two types only are feasible: essentially we can put $\beta_y = b_y/b_x = 0$ and take the remaining three parameters ϑ', ϑ'', e β_z as unknowns, or vice-versa set $\vartheta' = 0$ and the unknowns become ϑ'', β_y, e β_z. Similarly to photogrammetry, the two types of orientation could be called symmetric and asymmetric (fig. 6).

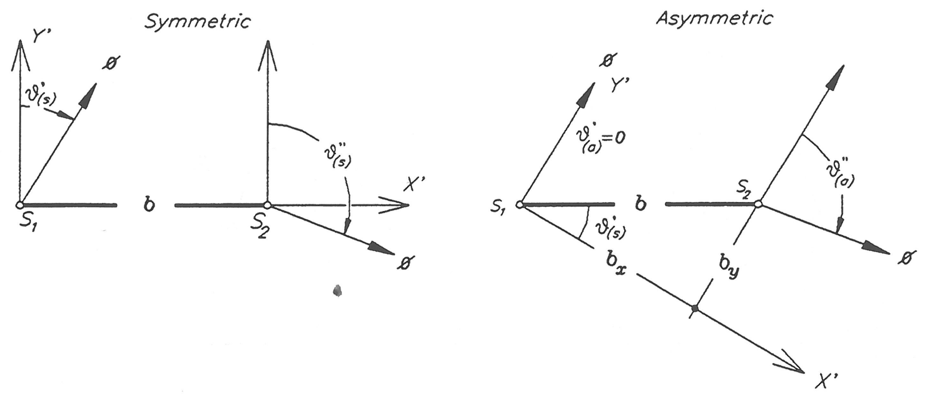


Fig. 6 - The symmetric relative orientation (left) and the asymmetric R.O. (right)

1. Symmetric Relative Orientation:

$$\vartheta'_{(S)},\quad \vartheta''_{(S)},\quad e\quad \beta_z$$

2. Asymmetric Relative Orientation

$$\vartheta''_{(a)},\quad \beta_y,\quad e\quad \beta_z$$

It is possible to pass from the parameters of the one type to the other one (fig.6). In fact it is:

$$\vartheta''_{(a)} = \vartheta'_{(S)} - \vartheta''_{(S)} \qquad b_x = b.\cos\vartheta'_{(S)}$$

$$b_y = b.sin\,\vartheta'_{(S)} \tag{21}$$

Obviously the solution is iterative. We need to supply approximate values of the parameters. In the performed tests the convergence was achieved for initial values different by 30° from the final values (fig.11) also. The initial value of the normalised vertical component was set $\beta_z = 0$.

The procedure appears then useful mainly in local triangulations where there are large excursions in the vertical directions, as in the case of the determination of the control points for close-range photogrammetry. In this case in addition it is not necessary a universal reference system but only a locale one.

One could think that the instability condition is when all the points lie on the same plane. As matter of fact, the test Mire was prepared just to demonstrate the critical condition, arranging a set of the targets and the instrumental centres in the same horizontal plane. Indeed we got the best results, the convergence with the largest difference of the approximated values for the parameters, from their final values. Paradoxically we get the best conditionumber. Further investigations must be carried out, but up to now we can say that with respect to the analogue photogrammetric problem, the orientation for coplanarity in surveying is less sensitive to the critical instability conditions (see par. 5).

3. COMPUTATION OF THE RELATIVE CO-ORDINATES

The relative co-ordinates of an intersected point P (X'_P, Y'_P, Z'_P) are derived by solving the two straight lines equations (fig. 1):

$$\begin{cases} X'_P - Y'_P \cdot tan(\vartheta' + l') = 0 \\ X'_P - Y'_P \cdot tan(\vartheta'' + l'') = b_x - b_y \cdot tan(\vartheta'' + l'') \end{cases} \tag{22}$$

and as average of the two elevations coming from the two station points

$$Z'_P = \frac{Z_1 + Z_2}{2} \tag{23}$$

where

$$Z_1 = d_1 \cdot cotg\varphi'; \qquad Z_2 = d_2 \cdot cotg\varphi'' + b_z \tag{24}$$

being d_1 and d_2 the horizontal distances of P from S_1 and S_2, computed with the plane co-ordinates.
Equs.(22) and (24) can be linearised as function of the unknown co-ordinates X'_P, Y'_P, Z'_P, of the point P.

$$\begin{bmatrix} 1 & -\tan a_i & 0 \\ -\frac{(X_P^0 - X_S)}{d_0} cotg\varphi_i & -\frac{(Y_P^0 - Y_S)}{d_0} cotg\varphi_i & 1 \end{bmatrix} \cdot \begin{bmatrix} dX_P \\ dY_P \\ dZ_P \end{bmatrix} = \begin{bmatrix} -g_X^0 \\ -g_Z^0 \end{bmatrix} \tag{25}$$

having set

$$\begin{aligned} g_X &= X'_P - Y'_P \tan a_i - X'_{Si} + Y'_{Si} \tan a_i \\ g_Z &= Z'_P - d \cdot cotg\varphi_i - Z'_{Si} \end{aligned} \tag{26}$$

with $X'_{Si}, Y'_{Si}, Z'_{Si}$ the relative co-ordinates of the i.th station. The difference between the elevations Z_1 and Z_2 is similar to the photogrammetric transversal parallax p_y.

The analogy can be extended to the bad formation of the "model", the so-called model deformation with a remarkable difference: the worst deformation in photogrammetry derives from the bad solution of the relative rotation ω around the taking base. In this case such a possibility do not exist because ω=0 , for the nature itself of the angular measurements.

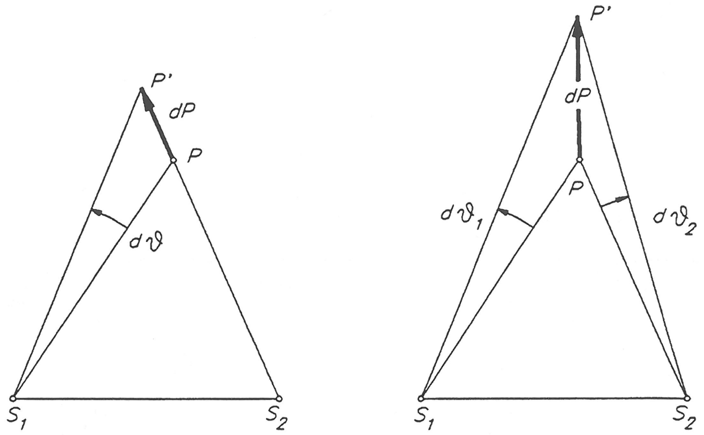
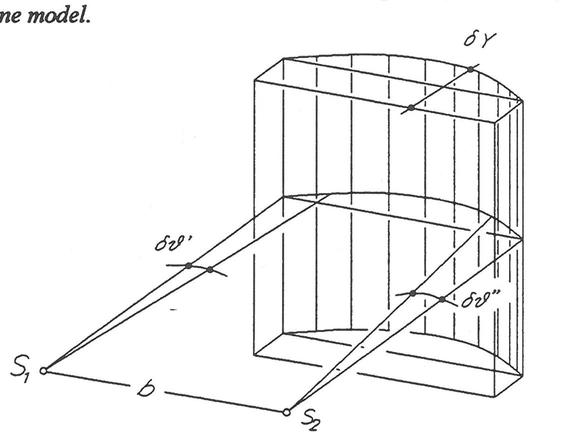


Fig. 7- Errors dP in the estimate of the co-ordinates of P due to an orientation error dϑ.

The only worried deformation derives then from the bad solution of the rotations transversal to the base ϑ' and ϑ'' that results in an error , $\delta\vec{P} = \delta X\vec{i} + \delta Y\vec{j} + \delta Z\vec{k}$ with $\vec{i}, \vec{j},$ *and* $\vec{k}$ versors of the co-ordinate axes. In the aerophotogrammetric model, for almost flat terrain, the larger component is the one in direction of the depth, that is the direction orthogonal to the base that in this case, corresponds to the $\vec{Y}$. [2]

Fig. 8 - Errors in the estimate of the co-ordinates caused by orientation errors δϑ (second order component) in case of plane model.



[2] In a equation system Ax = b where A is real, symmetric of order n, let be $Av_i = \lambda_i v_i$ where v_i è the i.th eigenvector of A and λ_i its associate eigenvalue . Let's put in increasing order the eigenvalues of A :

$$0 \le |\lambda_1| \le |\lambda_2| \le \ldots \le |\lambda_n| .$$

The conditionumber of A , indicated with *cond (A)* is defined as : $cond(A) = \dfrac{|\lambda_n|}{|\lambda_1|}$.

Obviously $cond(A) \ge 1$ for each A.; $cond(A) = 1$ where A is orthogonal. The matrix A is illconditioned when $cond(A) >> 1$ (es.10^3). When A is singular $\lambda_1 = 0$ and then $cond(A) = \infty$. In general, the larger the conditionumber, the worse is the solution of the system .

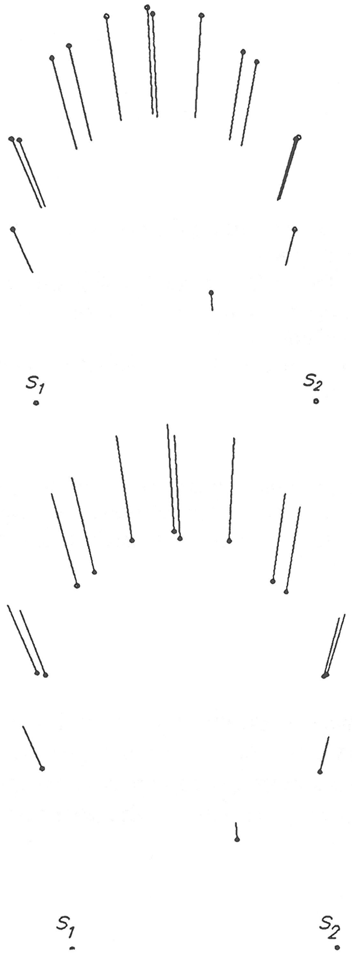


Fig. 9 a- Deformation from an orientation error dϑ= 1 g in the file Fontana the absolute orientation has the station points S_1 and S_2 as control points.

We consider here the case of the almost flat model only, as can be the one given by the facade of a building. As a matter of fact the most frequent application for the type of the here-described orientation is just this one. Then we can assess that

- the major component of the deformation is just in orthogonal direction to the base;

4. ABSOLUTE ORIENTATION

The model co-ordinates are transformed by means the well known similitude transformation, that is a roto-translation with scale variation with 5 parameters:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \lambda \begin{bmatrix} \cos K & -sinK \\ sinK & \cos K \end{bmatrix} \cdot \begin{bmatrix} X' \\ Y' \end{bmatrix} + \begin{bmatrix} X_{S1} \\ Y_{S1} \end{bmatrix} \quad (28)$$

having set

$$a = \lambda \cdot \cos K; \quad b = -\lambda \cdot sinK \quad (29)$$

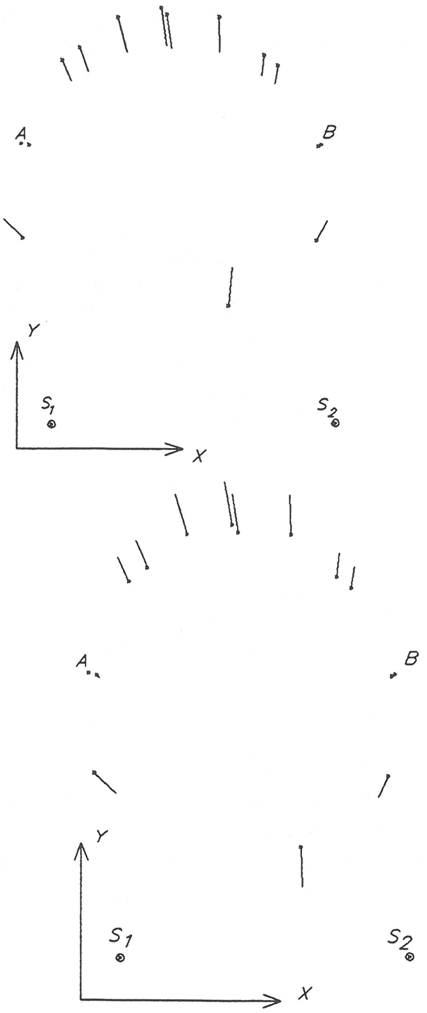


Fig. 9b -. The points A and B are the control points: the roto-translation has taken place from the relative system to the absolute system.

- that such a component is given by the sum of a linear term and by a second order term in the three variables X,Y and Z.

An appropriate absolute orientation (see next par. 4, 5 and 7) can absorb the linear component of the errors.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \lambda \cdot \begin{bmatrix} \cos K & -sinK & 0 \\ sinK & \cos K & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} \quad (27)$$

The transformation of the co-ordinates of the a point $P(X,Y,Z)$ occurs in two separated phases:

- planimetry (4 parameters: 1 rotation K, 1 coefficient of scale λ, 2 translations X_{S1}, Y_{S1}) -
- altimetry (1 translation Z_0)

$$[Z] = \lambda \cdot [Z'] + [Z_{S1}] \quad (30)$$

The estimate of the parameters of the (28) can be solved in a least squares procedure when the known points are 3 or more. Equs. (28) are written, reordering with respect to the unknown parameters:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X' & Y' & 1 & 0 \\ Y' & -X' & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ X_{S1} \\ Y_{S1} \end{bmatrix} \qquad (31)$$

At least two points in the final system known in planimetry and a point known in elevation are needed. In particular we can fix the exterior absolute reference system passing through two points whose horizontal distance only has been measured with a metric tape, having set the origin in the first point and the abscissa of the second point equal to the measured distance, procedure particularly simple in the case of architectonic survey. Among the known points are included the theodolite stations also.

Opposite to the A.O. of the photogrammetric model, no iteration is needed, because the equations (28) and (30) are linear in the unknowns.

An appropriate choice of the reference points can bring to satisfying results.

4.1 The effect of sphericity and refraction

In order to impose the coplanarity of the four points S_1 , S_2 , P' and P" it is necessary to correct previously the fictitious plate co-ordinates. Such a correction in fact is (fig.10):

$$\delta z = \frac{\delta\varphi . c}{sin^2\varphi} = \frac{\lambda d}{2Rsin^2\varphi} \qquad (32)$$

derived by differentiating the $z = c.\operatorname{cotg}\varphi$ and by substituting $\delta\varphi = d/2R$, with R radius of the local sphere. The correction (32) requires on the contrary the preventive knowledge of the scale coefficient λ, which is of the same order of magnitude of d . (For $\varphi = 90°$ and $\lambda \approx d$ we get the well-known expression of the sphericity effect).

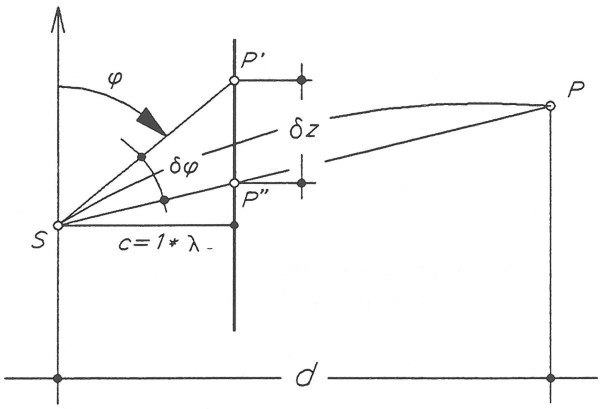


Fig.10- Corrections for refraction to the fictitious plate co-ordinate z

If the distances of the observed point P from the station points S and S are equal, we can neglect from such a correction to impose the coplanarity, except afterwards take into account the effect of the sphericity and refraction

$$\frac{(1-k)}{2R}d^2 \qquad (33)$$

in the absolute orientation where we can use the (22) where the distances are already at terrain scale and R is the radius of the local sphere and k is the coefficient of refraction.

$$Z_1 = d_1 \cdot cotg\varphi' + \frac{(1-k)}{2R}d_1^2 \quad ; \quad Z_2 = d_2 \cdot cotg\varphi'' + b_z + \frac{(1-k)}{2R}d_2^2 \qquad (34)$$

5. PRACTICAL ESPERIENCES

Seven surveys are here presented. Experiences of computation have been made either with data file already available either with expressly prepared observations. All the surveys have been previously adjusted to least squares with the programme RETE and with the three-dimensional adjustment with the programme 3D, except the one called M.Sicuro test. The co-ordinates have been then compared with those derived by coplanarity. In table 1 are shown the values RMS of the differences in X, Y and Z.

- The experiences nn. 1, 2 and 6 named Ss.Annunziata, Fontana and Spoleto refer to close-range triangulations to short distance for the architectural projects.
- The test n. 5 M.Sicuro is taken from a triangulation work for the photogrammetric control for an aerophotogrammetric 1/1000 scale map. The distances were in the order of few kilometres A similar case, non-present in the table, was not successful but the distances were much larger.
- The tests nn. 3 and 4, called Mire and Cerchio, have been expressly prepared to test the programme.

- The cases nn. 6 and 7, Spoleto and Diga are relative to the triple intersection also. For an architectural survey of the facade of a building in Spoleto 22 points have been signalised and then intersected from three stations. In a dike the control network is formed by a series of 15 signalised points. The measures of angles and distances are from three stations S_1,S_2, and S_3. These results are described and commented in 8. [3] We have taken into consideration here the simple intersection from the two most external stations.

In order to check whether the errors of the co-ordinates are significantly different from zero, these errors have been tested. For every case the sample mean s.d. $\sigma_{\overline{X}}$ has been derived from the s.d. σ_X of the co-ordinates traditionally derived. $\sigma_{\overline{X}} = \frac{\sigma_X}{\sqrt{n-1}}$. The errors have been normalised $t = \frac{dX}{\sigma_{\overline{X}}}$ and the normalised value has been compared with the critical value $t^*_{0.05}$ taken from the tables of the t of Student with a level of significance equal to the 5%. If $t < t^*_{0.05}$ the null hypothesis H_0 holds and that is $E\{t\} = 0$; in the contrary case the alternative hypothesis H_1 holds, $E\{t\} > 0$.

In the graph fig 11 the number of iterations needed to the convergence is shown in the 7 studied cases for increasing values of the difference between the value approximate initial of orientation for the two stations and the one final ones. The angles are expressed in grades. The graph is interrupted in correspondence of the abortion.

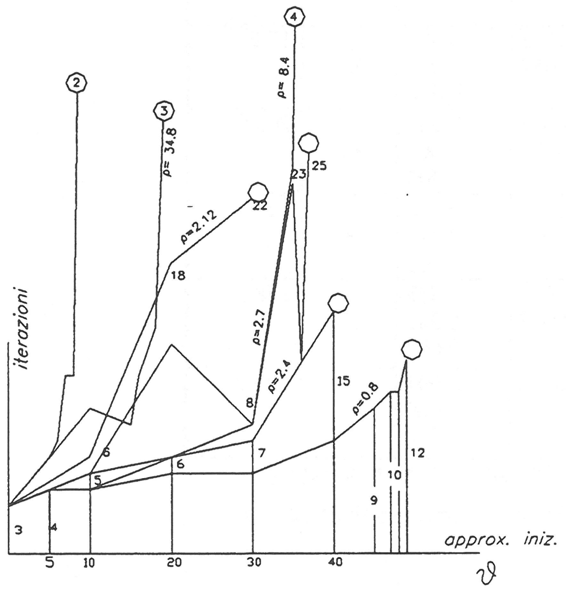


Fig. 11 - Number of needed iterations as function of the approximated values of the orientations -

In order to demonstrate the importance of a convenient choice of the points with whom make the Absolute Orientation, we made the following test:

in the case 2) Fontana, the 15 points are placed along the walls nearly cylindrical. Errors of 1 degree in the orientation of the S_1 and of -1 g in the S_2 have been simulated: first the A.O. has been carried out utilising the terrain co-ordinates of the points of station. The behaviour of the errors is the one shown in fig. 9a. Afterwards the A.O. has been made with the control points A and B. The final errors are very much reduced (fig. 9b). It still remains nevertheless a systematic error.

Similarly we operate in the Diga test, simulating an orientation error of opposite sign in the two external stations (fig. 19). If the A.O. is around the base S_1S_2, we have a remarkable systematism. When on the contrary the A.O. takes place with A and B control points, the errors are reduced to non-linear part, while the errors are transferred to the station points.

As already mentioned, to verify the instability of computation, 29 signals have been placed on the 4 walls of the laboratory with the help of a spirit level. The two theodolite stations had the instrumental centre on the same horizontal plane of a series of signals. In this way we thought to create the conditions of indetermination for the computation of orientation. Although this, the computation was successful with good accuracy, (table 1). With respect to the similar photogrammetric problem, the orientation in surveying for coplanarity seems then to be less sensitive to the critical conditions of numerical instability.

5.1 - Comparison by means of the non-parametric tests

The following tests have been carried out to check the correspondence of the medians of the distributions obtained with the coplanarity and the classical method of intersection. The co-ordinate differences have been computed. The non-parametric tests should be more suitable for non normal distributions distributions. On the other end the reduced amount of samples doesn't allow to use the Pearson test to verify the normality of their distribution. Where the sample is consistent the Pearson test was applied. It follows the scheme of the tests.

In the table 4 the results of the tests are summarised.

[3]The two last data files have been supplied by ing. Delfo Palpacelli , Macerata

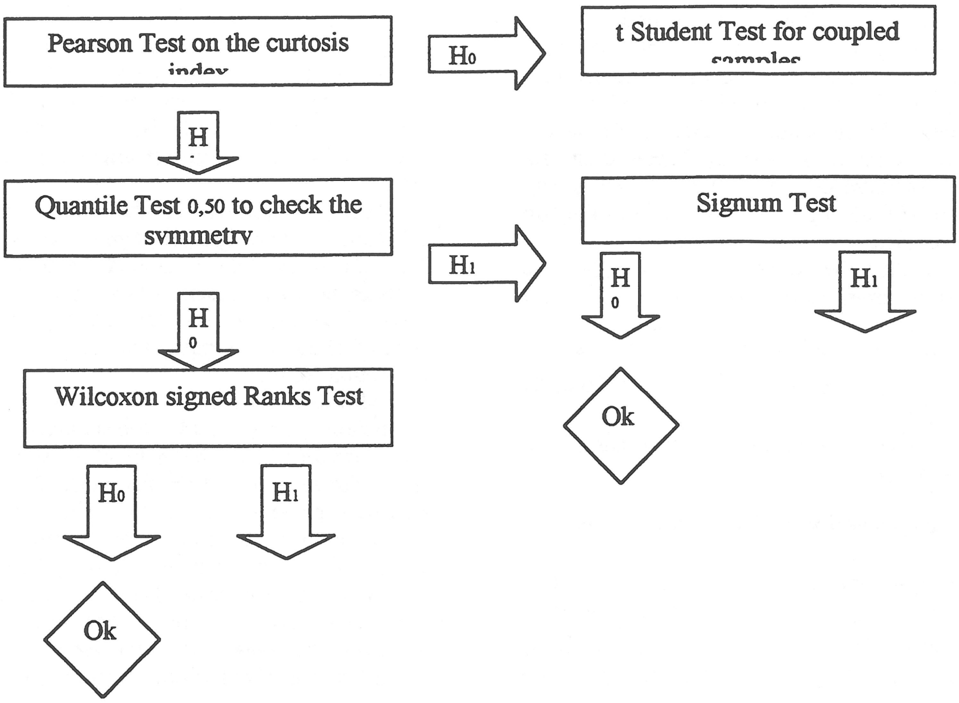


Table 1 - Double Intersection - Results

N	Work	n.pun	estim. dir (g)	estim .dir. (g)	obser. Dir. (. g)	dteta (g)	dX,dY, dZ. (m)	test	cond(A)
1	Ssannun	14	99.6647 298.3049	±0.0089 ±0.0101	99.6789 298.3170	+0.0142 +0.0121	0.008 0.004 0.007	H_0	256.0
2	Fontana (*)	15	100.0142 299.7149	±0.0453 ±0.0453	99.9985 299.7450	-0.0157 +0.0301	0.001 0.001 0.001	H_0	112.8
3	Mire (*)	29	190.9140 357.7415	±0.0024 ±0.0019	190.9117 357.7361	-0.0023 -0.0054	0.001 0.001 0.000	H_0	17.5
4	Cerchio(*)	7	318.5005 50.5684	±0.0166 ±0.0095	318.5093 50.5658	+0.0088 -0.0026	0.001 0.001 0.000	H_0	263.8
5	M.sicuro (***)	5 -	315.8845 39.8715	±0.0068 ±0.0248	315.8839 39.8792	-0.0006 +0.0077	0.035 0.011 0.065	H_0	6436
6	Spoleto(*) (**)	22	76.4851 273.9936	±0.0056 ±0.0071	76.4778 273.9889	-0.0073 -0.0047	0.003 0.001 0.002	H_0	278
7	Diga(*) (**)	15	99.9999 300.0012	±0.0017 ±0.0016	100.0001 300.0001	-0.0002 +0.0011	0.004 0.006 0.012	H_0	290.3

(*)Signalised Points

(**) Intersection from two stations only

(***) Large distances surveying

(3) Intersection from three stations

Table 2 – Results of the non parametric tests

File	N. pun.	Quanti le Test			Wilco xon Test			Signu m Test			Correction		
		0,50											
		□x	□y	□z	□x	□y	□z	□x	□y	□z	□x	□y	□z
Ssasunn	16	H0	H0	H1	H0	H0	/	/	/	H0	/	/	/
Fontana	17	H1	H0	H0	/	H0	H0	H0	/	/	/	/	/
Mire	29	H0	H0	H1	H0	H1	/	/	/	H1	/	0,002	-0,001
Cerchio	9	H0	H1	/	H0	/	/	/	H1	/	/	□□□□ □	/
Msicuro	5	H0	H0	H0	H1	H1	H1	/	/	/	-0,103	0,094	-0,03
Spoleto	22	H0	H0	H0	H1	H1	H1	/	/	/	0,003	-0,001	-0,002
Diga	15	H0	H0	H0	H1	H1	H0	/	/	/	-0,002	-0,011	/

These results have been compared with the results coming from the parametric statistics, in particular they have been compared with those get from t test for coupled tests, that is the equivalent parametric test. The table summarises the results:

Table 3- T Test with coupled samples

File	N. pun.	□x	□y	□z
Ssasunn	16	H0	H0	H0
Fontana	17	H0	H0	H0
Mire	29	H0	H1	H1
Cerchio	9	H0	H0	/
Msicuro	5	H0	H0	H0
Spoleto	22	H1	H1	H1
Diga	15	H1	H1	H0

As we can see, the results are essentially the same where the sample is sufficiently consistent (a dozen at least).

6. Indetermination in the plane resection

In the plane resection (Snellius- Pothenot problem) there is indetermination when the three observed points P_1, P_2, P_3 and the station S_1 lie on the same circle. Such an indetermination is coped by following the here described approach. In the experiences named Cerchio we verified that with the here described approach this indetermination do not take place. None of the two independent stations could be determined. Some targets have been placed on the floor along a circle and also the measuring stations have been placed along the circumference. Also the value of the conditionnumber, non-particularly high with respect to the other cases (264), do not show the possible illconditioning of the solving system.
That can be easily explained: in fact, while from any point of the circle the arcs are seen under the same angles, there is only one position that minimises the elevation differences.

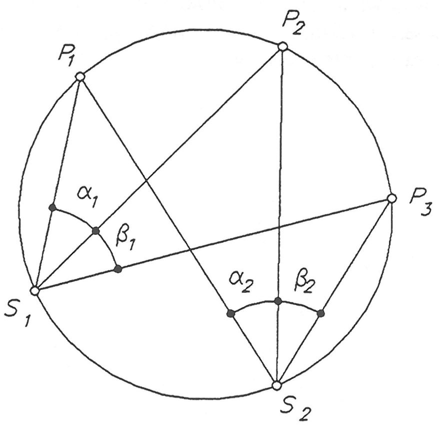


Fig. 12 - Case of indetermination in the resection

7. THE INTERSECTION FROM THREE STATIONS

In case of a triple intersection, say intersection of the generic point P from three stations S_1, S_2 and S_3 , the interested planed are three (fig.13). For any observed point we can then write three condition equations. The unknowns are 10 in this case, say :

- the three orientations angles

3

- the two components of one of the three bases
  2
- the three components of the remaining two bases 3x2=
  6

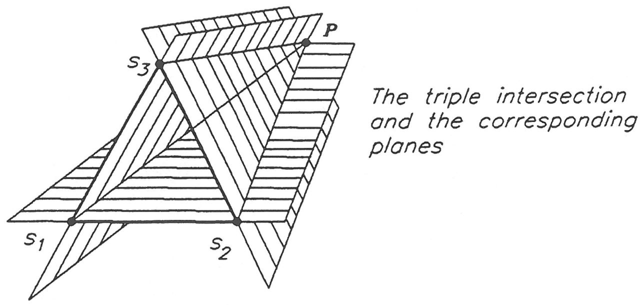


Fig. 13 - Triple intersection: the intersected planes

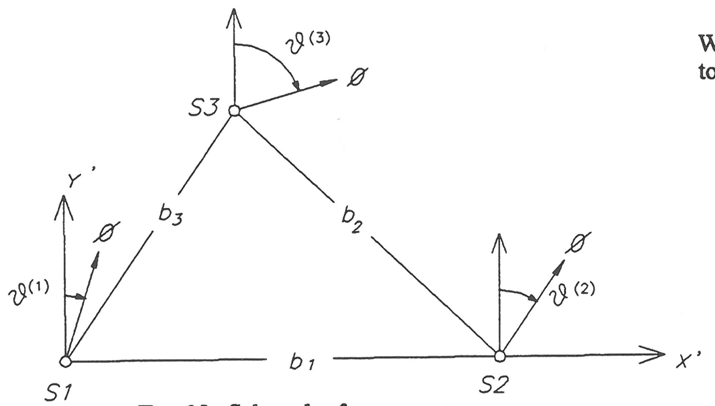


Fig. 15 - Selected reference system

Alternatively we can fix the reference system as in figs. 15 and 16. In order to find the initial values of the components of the bases provisional initial values are given to the co-ordinates of the three vertices.

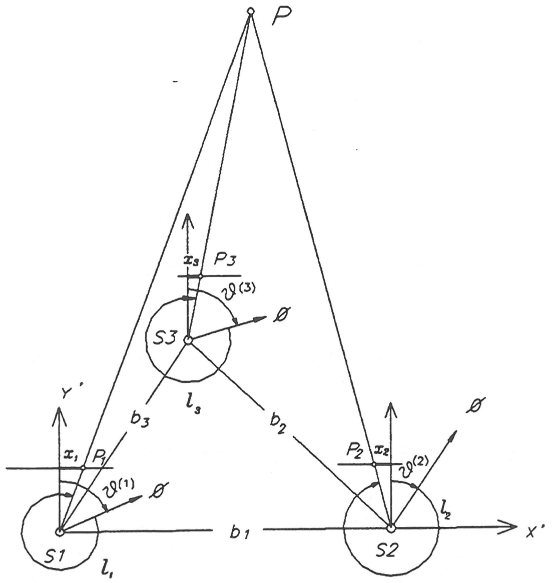


Fig. 16 - The three intersecting straight lines

Let's define a positive versus in the triangle sides. The components (with sign) of the three bases:

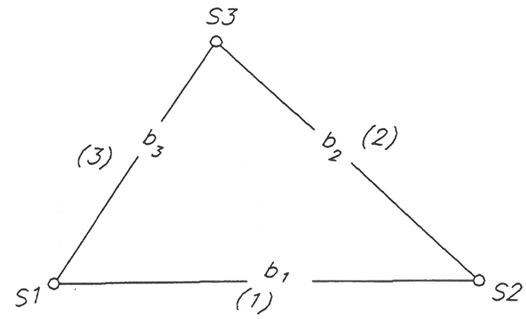


Fig. 14 - Triple Intersection : the three bases

We can select the asymmetric orientations and then set equal to zero the orientation of one of the three stations.

$$\begin{cases} b_x^{(i)} = X'(i+1) - X'(i) \\ b_y^{(i)} = Y'(i+1) - Y'(i) \\ b_z^{(i)} = Z'(i+1) - Z'(i) \end{cases} \quad (35)$$

Because of the chosen system, $b_x^{(1)} = 1$ and $b_x^{(1)} = 0$. To the coplanarity equations three conditions must be added, relative to the components in the three directions of the three bases.

$$\begin{cases} g_x = b_x^{(1)} + b_x^{(2)} + b_x^{(3)} = 0 \\ g_y = b_y^{(1)} + b_y^{(2)} + b_y^{(3)} = 0 \\ g_z = b_z^{(1)} + b_z^{(2)} + b_z^{(3)} = 0 \end{cases} \quad (36)$$

In reality not all the 10 parameters of orientation are independent: In fact set

$b_x^{(1)} = 1$ and $b_y^{(1)} = 0$ from the (36) we have $b_x^{(3)} = 1 - b_x^{(2)}$ $b_y^{(3)} = -b_y^{(2)}$ and

$$b_z^{(3)} = -(b_z^{(1)} + b_z^{(2)}) \quad (37)$$

Then the independent parameters are only the following seven:

ϑ_1, ϑ_2, ϑ_3, $b_z^{(1)}$, $b_x^{(2)}$, $b_y^{(2)}$, $b_z^{(2)}$,

and the three linearised coplanarity conditions are:

$$\begin{bmatrix} \frac{\partial f_1}{\partial \vartheta^{(1)}} & \frac{\partial f_1}{\partial \vartheta^{(2)}} & \cdot & \frac{\partial f_1}{\partial b_z^{(1)}} & \cdot & \cdot & \cdot \\ \cdot & \frac{\partial f_2}{\partial \vartheta^{(2)}} & \frac{\partial f_2}{\partial \vartheta^{(3)}} & \cdot & \frac{\partial f_2}{\partial b_x^{(2)}} & \frac{\partial f_2}{\partial b_y^{(2)}} & \frac{\partial f_2}{\partial b_z^{(2)}} \\ \frac{\partial f_3}{\partial \vartheta^{(1)}} & \cdot & \frac{\partial f_3}{\partial \vartheta^{(3)}} & \frac{\partial f_3}{\partial b_z^{(1)}} & \frac{\partial f_3}{\partial b_x^{(2)}} & \frac{\partial f_3}{\partial b_y^{(2)}} & \frac{\partial f_3}{\partial b_z^{(2)}} \end{bmatrix}_0 \cdot \begin{bmatrix} d\vartheta^{(1)} \\ d\vartheta^{(2)} \\ d\vartheta^{(3)} \\ db_z^{(1)} \\ db_x^{(2)} \\ db_y^{(2)} \\ db_z^{(2)} \end{bmatrix} = \begin{bmatrix} -f_0^{(1)} \\ -f_0^{(2)} \\ -f_0^{(3)} \end{bmatrix} \tag{38}$$

or :

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & a_{14} & \cdot & \cdot & \cdot \\ \cdot & a_{22} & a_{23} & \cdot & a_{25} & a_{26} & a_{27} \\ a_{31} & \cdot & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \end{bmatrix}_0 \cdot \begin{bmatrix} d\vartheta^{(1)} \\ d\vartheta^{(2)} \\ d\vartheta^{(3)} \\ db_z^{(1)} \\ db_x^{(2)} \\ db_y^{(2)} \\ db_z^{(2)} \end{bmatrix} = \begin{bmatrix} -f_0^{(1)} \\ -f_0^{(2)} \\ -f_0^{(3)} \end{bmatrix} \tag{39}$$

For the *i.th* base of ending points the *i* and *i+1* the coefficients from the (39) are:

$$\frac{\partial f_i}{\partial \vartheta_i} = \frac{1}{\cos^2(\vartheta_i + l_i)} \left[\frac{b_y^{(i)} tg(\vartheta_{i+1} + l_{i+1}) sin(\vartheta_i + l_i)}{tg\varphi_i} + b_z^{(i)} + \right.$$
$$\left. - \frac{b_x^{(i)} sin(\vartheta_i + l_i)}{tg\varphi_i} - \frac{b_y^{(i)}}{tg\varphi_{i+1} \cos(\vartheta_{i+1} + l_{i+1})} \right] = c_i \cdot \left[-t_i b_x^{(i)} + (x_{i+1} t_i - z_{i+1}) b_y^{(i)} + b_z^{(i)} \right]$$

$$\frac{\partial f_i}{\partial \vartheta_{i+1}} = \frac{1}{\cos^2(\vartheta_{i+1} + l_{i+1})} \left[- \frac{b_y^{(i)} tg(\vartheta_i + l_i) sin(\vartheta_{i+1} + l_{i+1})}{tg\varphi_{i+1}} - b_z^{(i)} + \right.$$
$$\left. \frac{b_x^{(i)} sin(\vartheta_{i+1} + l_{i+1})}{tg\varphi_{i+1}} + \frac{b_y^{(i)}}{tg\varphi_i \cos(\vartheta_i + l_i)} \right] = -c_{i+1} \cdot \left[-t_{i+1} b_x^{(i)} + (x_i t_{i+1} - z_i) b_y^{(i)} + b_z^{(i)} \right]$$

$$\frac{\partial f_i}{\partial b_x^{(i)}} = -\frac{1}{tg\varphi_i \cos(\vartheta_i + l_i)} + \frac{1}{tg\varphi_{i+1} \cos(\vartheta_{i+1} + l_{i+1})} = z_{i+1} - z_i$$

$$\frac{\partial f_i}{\partial b_y^{(i)}} = \frac{tg(\vartheta_{i+1} + l_{i+1})}{tg\varphi_i \cos(\vartheta_i + l_i)} - \frac{tg(\vartheta_i + l_i)}{tg\varphi_{i+1} \cos(\vartheta_{i+1} + l_{i+1})} = x_{i+1} z_i - x_i z_{i+1}$$

$$\frac{\partial f_i}{\partial b_z^{(i)}} = -tg(\vartheta_{i+1} + l_{i+1}) + tg(\vartheta_i + l_i) = x_i - x_{i+1} \tag{40}$$

having set:

$$a_i = \vartheta_i + l_i \qquad a_{i+1} = \vartheta_{i+1} + l_{i+1}$$

$$z_i = \frac{\cot g\varphi_i}{\cos a_i} \qquad z_{i+1} = \frac{\cot g\varphi_{i+1}}{\cos a_{i+1}}$$

$$x_i = \tan a_i \qquad x_{i+1} = \tan a_{i+1}$$

$$c_i = 1/\cos^2 a_i \qquad c_{i+1} = 1/\cos^2 a_{i+1}$$

$$t_i = sin a_i / \tan\varphi_i \qquad t_{i+1} = sin a_{i+1} / \tan\varphi_{i+1} . \tag{41}$$

The function (11) is written again:

$$f_i = (z_{i+1} - z_i)\cdot b_x^{(i)} + (x_{i+1}z_i - x_i z_{i+1})\cdot b_y^{(i)} + (x_i - x_{i+1})\cdot b_z^{(i)} = 0 \qquad (42)$$

The coplanarity condition relative to the third base, keeping in mind the (37), becomes:

$$f_3=(z_1-z_3)\cdot(-1-b_x^{(2)})+(x_1z_3-x_3z_1)\cdot b_y^{(2)}-(x_3-x_1)(b_z^{(1)}+b_z^{(2)})=0 \qquad (43)$$

Its partial derivatives are:

$$\frac{\partial f_3}{\partial b_z^{(1)}} = -(x_3 - x_1) \qquad \frac{\partial f_3}{\partial b_x^{(2)}} = -(z_1 - z_3)$$

$$\frac{\partial f_3}{\partial b_y^{(2)}} = -(x_1 z_3 - x_3 z_1)$$

$$\frac{\partial f_3}{\partial b_z^{(2)}} = -(x_3 - x_1) \qquad (44)$$

In the (38) and (39) all the non-indicated coefficients are equal to zero. To solve for the 7 unknowns three points at least must be observed, with a redundancy equal to 2 (in the case of intersection only from two stations the redundancy would be zero).

With *n* observed points (*m*= observ. dir., *c* =equat. ; *u*= unknowns ; *r* =redun.:)

from 2 stations: $m=2n$ $u=2$, $c=n$, $r=n-2$

$m=2(n-1)$

from 3 stations: $m=3n$ $u=7$, $c=3n$, $r=3n-7$

$m=(3n-7)$

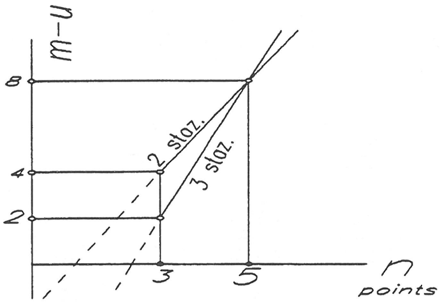


fig. 17- Number of the observed points and number of observations

From the graph in fig. 17 one can see that when $n \geq 5$ for increasing the redundancy it is convenient to set up three stations better than enlarge the number of the observed points from two stations.

8. Practical experiences. Example n. 6 (Spoleto) and n. 7 (Diga)

The tests ns. 6 and 7 have been carried out also with the triple intersection. For both tested cases have been carried out three computations of orientation:

A - simple intersection from the two most exterior stations

1. - with four points
2. - with all the available points

B - triple intersection with all the available points.

The results do not differ very much between them. The triple intersection improves the estimate of the orientations, nevertheless the absolute orientation makes the co-ordinates of the points not to be different. In fact also with the simple intersection, with the least number of the points (four) - case A1-, the residuals on the points are not practically different from those obtained with the triple intersection and with all the available points.

In fig. 19 on the left the absolute orientation was achieved by imposing the terrain co-ordinates of the stations S_1, S_2. On the right, on the contrary the points A and B have been utilised as control points for the absolute orientation . The errors on the 15 points are strongly reduced, whilst are increased on the station points S_1, S_2.

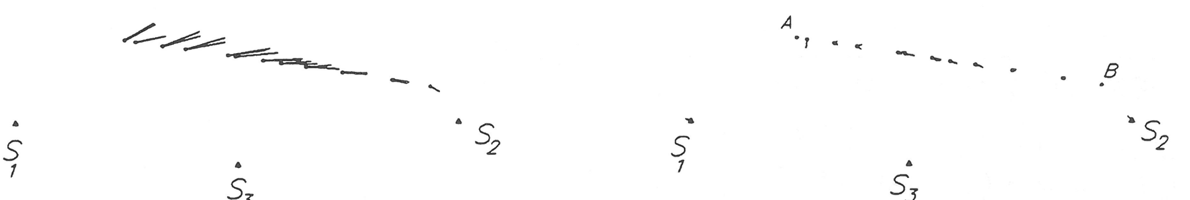


Fig. 18 - Spoleto Test. On the left the orientation with control points S1S2, on the right the control points are A and B. The errors are amplified 300 times.

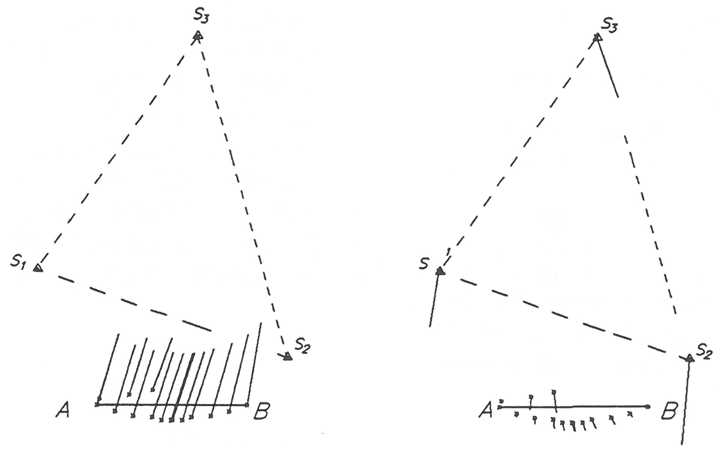


Fig.19 - Diga Test - Orientation error dϑ=1g. The errors are amplified 100 times. On the left the orientation as control points, the station points S_1 and S_2. On the right the control points are the points A and B.

Table 4 - Double intersection Vs. Triple intersection - Results

N	Work	n.pun	dir.stim. (g)	dir.stim. (g)	dir.osser. (g)	dteta (g)	dX,dY, dZ. (m)	test	cond(A)
A 1	Spoleto Two	4	76.4851 273.9936	±0.0056 ±0.0071	76.4778 273.9889	-0.0072 -0.0048	0.003 0.001 0.002	H_0	321
A 2	Spoleto Two	22	76.4851 273.9936	±0.0006 ±0.0031	76.4778 273.9889	-0.0072 -0.0048	0.005 0.002 0.002	H_0	278
B	Spoleto Three	22	76.4850 273.9937 99.8928	±0.0006 ±0.0018 ±0.0028	76.4778 273.9889 99.8933	-0.0072 -0.0048 +0.0005	0.001 0.001 0.001	H_0	150
A 1	Diga Two	4	99.9993 300.0009	±0.0017 ±0.0016	100.0001 300.0001	-0.0008 +0.0002	0.002 0.011 0.013	H_0	360
A 2	Diga Two	15	99.9999 300.0012	±0.0012 ±0.0008	100.0001 300.0001	-0.0008 +0.0002	0.004 0.006 0.012	H_0	290
B	Diga Three	15	100.0009 299.9999 160.6867	±0.0017 ±0.0016 ±0.0141	100.0001 300.0001 160.6923	-0.0008 +0.0002 +0.0056	0.002 0.003 0.001	H_0	165

(*)Signalised points

9. CONCLUSIONS

With the here described procedure the surveying operations are simplified for the intersection. In fact the needed measuring equipment is reduced to a theodolite, a tripod and a metric tape. One can avoid to connect the theolite stations, either with angular measurements or distances, the instrument height is not needed, the full traversing equipment is not needed as well. In all the tested cases we got an accuracy in the estimate of the bearings equivalent to its direct measure. Few points are sufficient for the orientation. The initial approximate bearings can be supplied with an approximation of some ten degrees. Only in one case the computation was not successful, the distances between the points were in the order of some kilometres. The procedure is mostly convenient for local triangulations. A good distribution of the observed points and a very good measure of the vertical angles are essential. Few points for the convergence of the computation of orientation are enough. With a good distribution of the control points for the absolute orientation, the results of the estimate of the co-ordinates of the surveyed points is absolutely equivalent to those of a traditional survey. The not perfect solution of the parameters of orientation brings to a systematic error in the estimate of the co-ordinates of the intersected points. The errors are reduced to acceptable values, say of the same order of magnitude of the s.d. of the classical intersection, when the absolute orientation takes place with well positioned control points. The most probable application of the described procedure is in the survey of control points for close-range photogrammetry.
The triple intersection increases very much the redundancy but do not improve further the quality of the results.

Acknowledgements
I wish to tank Gianlucva Gagliardini for having performed the non parametric tests.

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