ROBUST STATISTICAL METHODS FOR THE GPS NETWORK ANALYSIS

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ABSTRACT:

The possible presence of outliers in observations requires a refined statistical analysis for the traditional, as well as for GPS networks. After a least square adjustment is performed, traditional methods essentially consist in a standardised residual analysis. The so-called "preliminary diagnostic procedures" allow the identification of outliers and their removal. The "Baarda data snooping", the "τ-test" and the "Danish method" are well known statistical methods commonly applied in Geodesy. In the last few years a class of alternative estimation strategies has been proposed, known as robust methods (*Huber*, *Hampel*, *Rousseeuw*). A major advantage in the use of such methods is that the parameters obtained are only slightly influenced by outliers in the observations. Two different applications of robust estimation are discussed in the paper; their application to a simulated GPS network and to a real GPS net for deformation control is then presented.

1. INTRODUCTION

One of the most simple and widely used methods for the checking of small movements of soil and buildings is based on repeated observations, at different epochs, their subsequent treatment (numerical and statistical), and finally the comparison of the results of the various epochs.
The movements observed are often very small (e. g. in a landslide) and of the same order of the measurement errors.
In such a situation it is important to carry out a sophisticated statistical analysis, because possible gross errors in the model (mathematical and/or stochastic) can be considered as movements, leading to an incorrect interpretation of the phenomenon.
Historically the most used estimation criterion is the Least Squares (introduced by Gauss in the XIX), essentially because:

- the L.S. solution is easily calculated, as it is reduced to a problem of minimum constraint;
- it takes on a precise geometric interpretation: it minimises the Euclidean distance between the measurement vector and the vector of the estimated values;
- if the hypotheses model (e. g. the observations are normally distributed) correspond to the sample data, then the estimator is correct, efficient, sufficient and consistent. It is also the maximum likelihood estimator (i.e. it is the one with the lowest variance among the correct estimators).

However, if the observations, or part of them, do not follow the hypothesised model, for instance due to the presence of gross errors, then the reliability of this estimator rapidly decreases.
So, in terms of efficiency, the maximum likelihood estimators (obtained in the normal distribution case) can lead to large inefficiencies even in presence of small contaminations.
Those observations not following the assumptions of the model are indicated as outliers.
To analyse the effects these gross errors have on the classic estimate, one can take into consideration the simple linear regression model: in fact, as this model allows the functional relationship between just 2 variables, the data can be visualised on a 2D-scatter plot.
This is generally not possible in geodetic surveying problems, which normally involve multi-dimensional variables.
It is usually possible to highlight different situations where gross errors take on different positions in reference to the other data (Fig. 1.1):

2. OUTLIERS AND ROBUST ESTIMATION METHODS

In the presence of outliers, two possible solution strategies can be carried out:

- the first involves the application of a series of **preventive diagnostic procedures** aimed at identifying the outliers, after which these observations are removed, and finally, a traditional estimate is carried out.
  Two examples of preventive diagnostic procedures are the **Baarda Data Snooping** and the **Pope's τ test,** which are included in the statistical test category;
- The second strategy involves the application of **robust estimate procedures,** where the outliers are identified after the parameters have been

estimated, and they are those having the greatest standardised residuals.

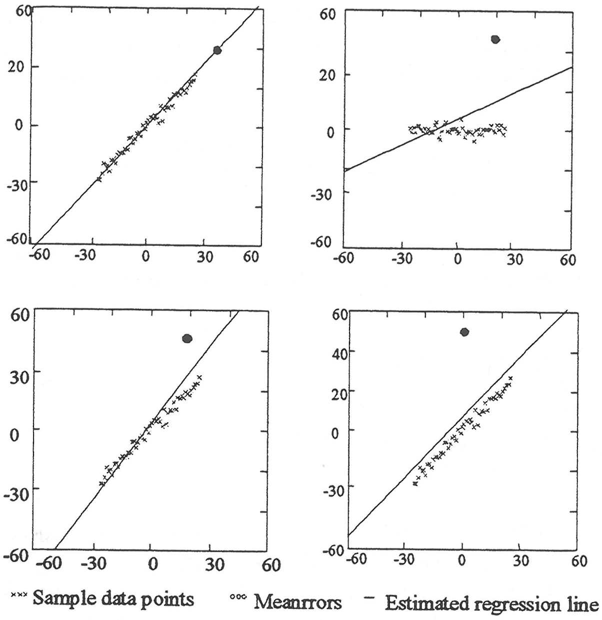


Fig. 1.1 Examples of gross errors in the data.

Therefore, diagnostic procedures and robust procedures really have the same goals, but have an opposite approach: when using diagnostic procedures, one first tries to delete the outliers and then to fit the "good" data by least squares, whereas a robust analysis first wants to fit a regression to the majority of the data and then to discover the outliers as those points which possess large residuals from that robust solution.
In this way, in addition to the classical properties of the estimators, robustness is introduced and used here as defined by the main authors of these:

"In a broad informal sense, robust statistics is a body of knowledge, partly formalised into "theories of robustness", relating to deviations from idealised assumptions in statistics" (Huber, 1977).

"Robust statistics, as a collection of related theories, is the statistics of approximate parametric models" (Hampel, 1986).

So, in short, robustness means the ability to estimate non-distorted parameters even in the presence of any outliers.
The main aims of robust statistics are:

- to describe the structure best fitting the bulk of the data;
- to identify deviating data points (outliers) or deviating structures for further treatment, if desired.

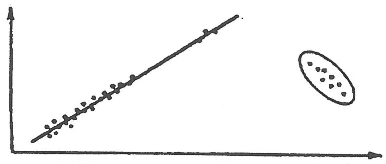


Fig. 2.1 The aim of robust statistics

3. HUBER'S MINIMAX APPROACH

The first approach to modern robustness theory was lead by P. Huber, in his 1964 paper. He introduced the **gross-error model**: instead of considering a precise distribution model F_o (which in general was the normal distribution in classical estimation), he fixed a kind of "neighbourhood" P_ε of the of model distribution F, and guaranteed optimal performance in that neighbourhood. It is defined by:

$$F(x-\theta) = (1-\varepsilon)\cdot F_0(x-\theta) + \varepsilon \cdot H(x-\theta)$$
$$F = (1-\varepsilon)\cdot \Phi + \varepsilon \cdot H$$
$$P_\varepsilon(F_0) = \{F_\varepsilon : F_\varepsilon = (1-\varepsilon)F_0 + \varepsilon H\}$$

The next step was the search for the **least favourable distribution** f_* in this neighbourhood, defined as the distribution by which the asymptotic variance V(T,F) of an estimator takes the highest value of the minimum:

$$L_F\left\{\sqrt{n}[T_n - T(F)]\right\} \underset{n\to\infty}{\rightarrow} N[0, V(T,F)]$$
$$v_1(\varepsilon) = \sup_{F\in P_\varepsilon} V(T,F) \quad \text{max asymptotic variance}$$
$$f_* : v_1(\varepsilon) = \min$$

After this, the estimator is chosen, which is the maximum likelihood estimator for the f_* function.
Therefore, by this approach, if the observations came from f*, then that estimator is the most efficient. If, however, they come from another distribution $\in P_\varepsilon$ then the asymptotic variance is less.
Huber's approach is called "minimax" because it minimises the maximum loss of adherence to the assumed model.

4. HAMPEL'S APPROACH OF ROBUSTNESS : THE INFLUENCE FUNCTION (1968,1974)

Hampel's contribution to the robustness theory can be summarised in three fundamental concepts: the qualitative robustness, the influence function and the breakdown point.

- an estimator is **qualitatively robust** if, having taken two distributions F and G "near" each other, then the distributions of T_n under F and G respectively remain near each other. Qualitative robustness therefore expresses the equicontinuity of the distributions of T_n with respect to n:

$$\forall \varepsilon > 0 \exists \delta > 0, n_0 > 0:$$
$$\forall G, \forall n \geq n_0, d_*(F,G) < \delta$$
$$\Rightarrow d_*[\mathrm{L}_F(T_n), \mathrm{L}_G(T_n)] < \varepsilon$$

- the **influence function** measures the effect, on the estimate, of an infinitesimal contamination of the F distribution at the point x, standardised by the mass of the contamination. Thus, it describes the asymptotic bias caused by contamination in the observations.

$$IF(x;T,F) = \lim_{t \to 0} \frac{T[(1-t)F + t\Delta_x] - T(F)}{t}$$

- the **breakdown point** expresses the greatest percentage of gross errors that can be tolerated in the sample by the estimator before that sample turns out to be not informative.

$$F(x-\theta) = (1-\varepsilon)\cdot G(x-\theta) + \varepsilon \cdot H(x-\theta)$$

The influence function describes three properties of the local robustness of an estimator:

- the **gross error sensitivity** γ, which measures the maximum bias that an infinitesimal contamination can cause to the estimated value of an estimator. If γ is finite, the estimator is called "β-robust";
- the **local shift sensibility** λ, which measures the estimator stability when replacing x by y, with x and y near each other;
- the **rejection point** ρ, which represents the shortest distance from the location parameter outside of which the observations do not contribute to the estimate value.

Therefore an estimator must:

- possess a γ value which is still quite low;
- be much more efficient than the median;
- possess a low local shift sensibility λ;
- possess a finite rejection point ρ.

5. CLASSES OF ESTIMATORS

The classification of robust estimators is very wide, and frequently an estimator can belong to more than one class because it can be obtained in different ways. The most common classes of estimators are:
M, L, A, P, S and W.
Among the various estimators analysed, only those used in the applications are given.

6. M ESTIMATORS (MAXIMUM LIKELIHOOD ESTIMATORS)

The M estimates derive from "generalised maximum likelihood"; in fact, these estimates come from a generalisation of the Maximum Likelihood Principle:

$$\sum_{i=1}^{n} -\ln f(\mathrm{T_n}; x_i) = \min \quad \text{or} \quad \sum_{i=1}^{n} \frac{f'(\mathrm{T_n}; x_i)}{f(\mathrm{T_n}; x_i)} = 0$$

In practice an M estimator is defined by a minimization problem of the form (ρ is the objective function):

$$\sum_{i=1}^{n} \rho(x\,; \mathrm{T_n}) = \min \quad \text{or}$$

$$\sum_{i=1}^{n} \psi(x_i; \mathrm{T_n}) = 0 \quad \text{if } \psi(x_i; \mathrm{T_n}) = \frac{\partial \rho(x_i; \mathrm{T_n})}{\partial \mathrm{T_n}}$$

but where the ρ function does not necessarily coincide with the logarithm of the density function f.
These two expressions supply the definition of the M estimator, and the nature of the ρ and Ψ functions determines the estimator properties.
The M estimates are more robust than the minimum norm (L_1) and are also robust against outliers in y direction.
However, their breakdown point is $\varepsilon = 1/n$, because the estimates are affected by outliers in the x direction. To solve this problem, redescending Ψ functions were introduced, so that they are annulled outside a certain interval.
Besides, as the M estimate cannot usually be explicitly calculated, in practice iterative calculus procedures are used, for example W estimate.
In the following applications, W estimators with redescending Ψ functions are mostly used. All of the estimators used in the present work are implemented in a Mathcad programme.
The design structure diagram and the Mathcad algorithm are shown in Fig. 6.2, both for W estimators and Siegel's repeated median estimator.

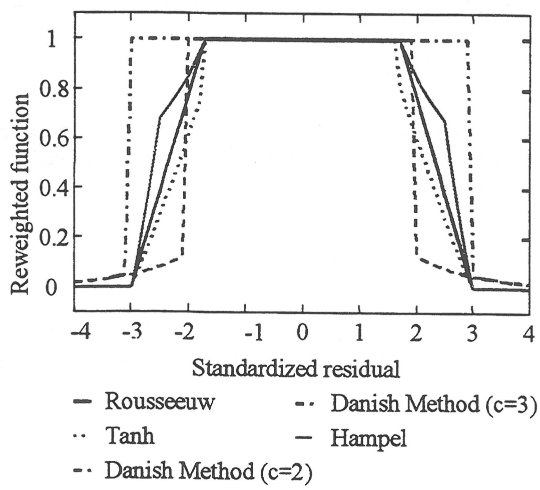


Fig. 6.1 Reweighted function

```
rob(A,P,l,c_2,max) := | vt←l
                      | Q_l←P^-1
                      | xt_cols(A)←0
                      | iter←1
                      | while iter≤max
                      |   | N_i←(A^T·P·A)^-1
                      |   | x←N_i·A^T·P·l
                      |   | v←A·x - l
                      |   | Q_v←(Q_l - A·N_i·A^T)
                      |   | s_o←sqrt(| v^T·P·v / tr(Q_v·P) |)
                      |   | j←1
                      |   | for i∈1..rows(A)
                      |   |   | s_i←sqrt(|Q_v_i,i|)·s_o
                      |   |   | α_i←| 1  if |v_i/s_i| < c_2
                      |   |   |     | e^-(|v_i/s_i|)  otherwise
                      |   |   | Q_l_i,i←Q_l_i,i / sqrt(α_i)
                      |   |   | P←Q_l^-1
                      |   |   | r_i←|v_i/s_i|
                      |   | vt←augment(vt,r)
                      |   | xt←augment(xt,x)
                      |   | iter←iter + 1
                      | usc_2←submatrix(vt,1,rows(vt),2,cols(vt))
                      | usc_3←submatrix(xt,1,rows(xt),2,cols(xt))
                      | usc_1←P
                      | usc
```

Fig. 6.2 W Estimators (Mathcad algorithm)

```
siegel(x,y,N) := | for i∈0..N
                 |   | k←0
                 |   | for j∈0..N
                 |   |   if i≠j
                 |   |     | vet_med_k←(y_j - y_i)/(x_j - x_i)
                 |   |     | k←k + 1
                 |   | med_j_i←median(vet_med)
                 | median(med_j)
```

Fig. 6.3 Siegel Estimators (Mathcad algorithm)

7. NETWORK LEVELLING FOR A DAM

Robust theories were first applied to data obtained from the testing of a dam by means of spirit-levelling network.

In this case, we are interested in the heights of a levelling network points on a dam at 36 epochs

Of the various points analysed, the results of point N° 519, which was situated on the crowning of the dam, are shown.

At the beginning, two different situations were considered: the set of observations were contaminated respectively by two "large" gross errors (about 5σ) and 2 "small" gross errors (about 2 σ).

Hypothesising a linear trend for the observations, there are two parameters to be estimated: the slope and the intercept of the regression straight line.

It was determined:

- an estimate according to classic criterion (L.S.);
- a robust estimate according to Siegel's repeated median estimator (sketched line)

$$\hat{\theta}_1 = \underset{i}{med}\,\underset{j \neq i}{med}\frac{y_j - y_i}{x_j - x_i}\ ,\ \hat{\theta}_2 = \underset{i}{med}\left(y_i - \hat{\theta}_1 \cdot x_i\right)$$

- the LS estimate excluding the first two observations, i.e. the contaminated ones (sketched line)

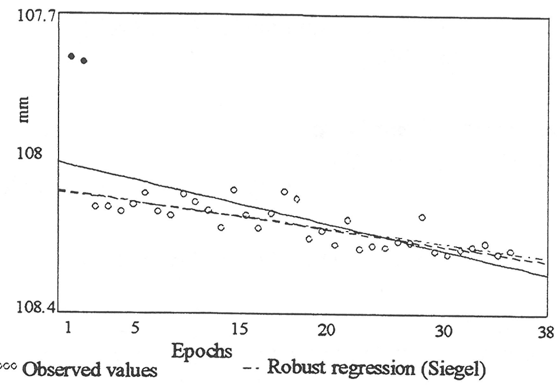


Fig. 7.1 Contamination with 2 "large" gross errors (~5σ)

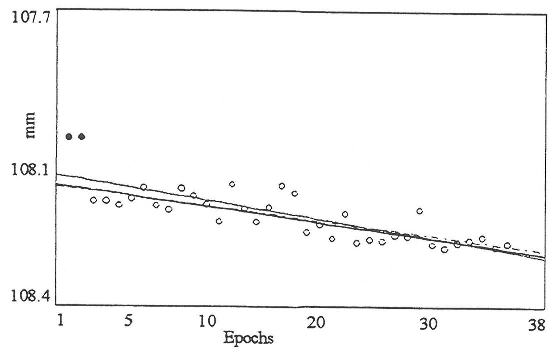


Fig. 7.2 Contamination with 2 "small" gross errors (~2σ)

From the results it can be seen that:

- In both cases the L.S. solution is highly affected by the presence of the two outliers, which tend to "attract" the regression straight line.
- On the other hand in both cases the robust solution almost coincides with the L.S. solution without contamination. This underlines the great efficiency of this estimator.

After this, two diagnostic indicators (Baarda's and Pope's) were applied to both the above situations to detect any gross errors.

Once the L.S. model was fixed, we tested the H_o null hypothesis "The model is correct both in the functional and in the stochastic part" versus the H_1 alternative hypothesis: "Only one outlier is present".

For Baarda's test, once a α_0 =20% for the single residue was fixed, and a β_o=20%, the threshold value was obtained, to be compared with the values of the standardised residuals of the observations.

In Pope's test, the threshold was obtained on the basis of the redundancy and after choosing an α of 5%.

Baarda's test (*Baarda data snooping*):

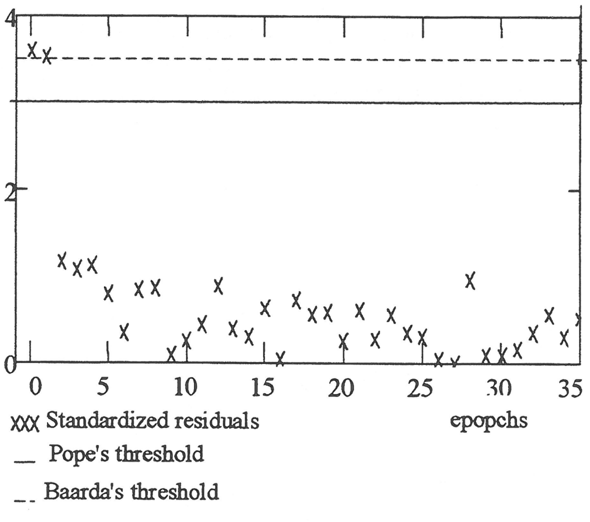
$$\left.\begin{array}{l}\alpha_0 = 20\% \\ \beta_0 = 20\%\end{array}\right| \rightarrow \begin{array}{l}u_{\alpha_0} = 3.5 \text{ (threshold value)} \\ \alpha \cong 20\%\end{array}$$

$$u_i = \frac{v_i}{s_{v_i}} = \frac{v_i}{s_0 \cdot \sqrt{q_{v_i v_i}}} \text{ (standardized residuals)}$$

Pope's test (*τ's method*):

$$\left.\begin{array}{l}\alpha = 5\% \rightarrow \alpha_0 = 0.1\% \\ r = 34\end{array}\right| \rightarrow \tau_{r,\alpha_0/2} = 2.98 \text{ (threshold value)}$$

$$u_i = \frac{v_i}{s_{v_i}} = \frac{v_i}{s_0 \cdot \sqrt{q_{\bar{v}_i v_i}}} \text{ (standardized residuals)}$$



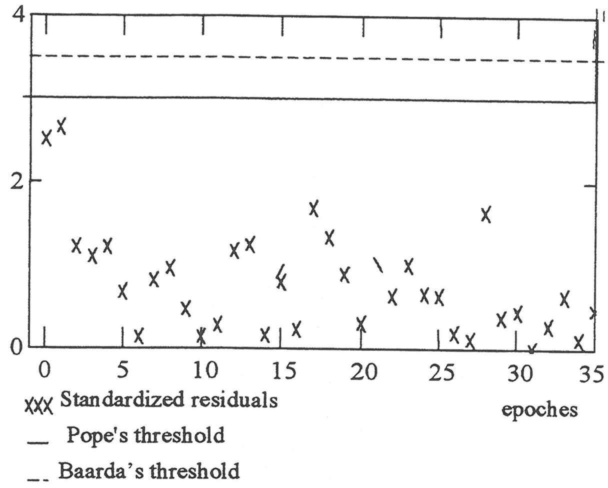


Fig. 7.3 Baarda and Pope's test

From the results obtained, it can be seen how in the first situation both tests are successful because they are able to pick out the two gross errors. However, in the second situation the tests fail.

This leads to the following observations:

- In Baarda's test an excessively high α value had to be assumed to highlight the errors inserted;
- The results of both tests depend on the number and entity of the gross errors;
- On the contrary, the robust estimate is reliable in both situations.

Given that the two tests cannot pick out the gross errors when they are quite small, this situation was analysed. However, this time we inserted four small gross errors (~ 2 σ).

A series of four robust estimators were applied to the sample data:

- the repeated median estimator (Siegel);
- Rousseeuw estimator;
- the Danish method;
- the weighted least squares solution;

The results are shown in the figure below and compared with the L.S. estimate without the four gross errors.

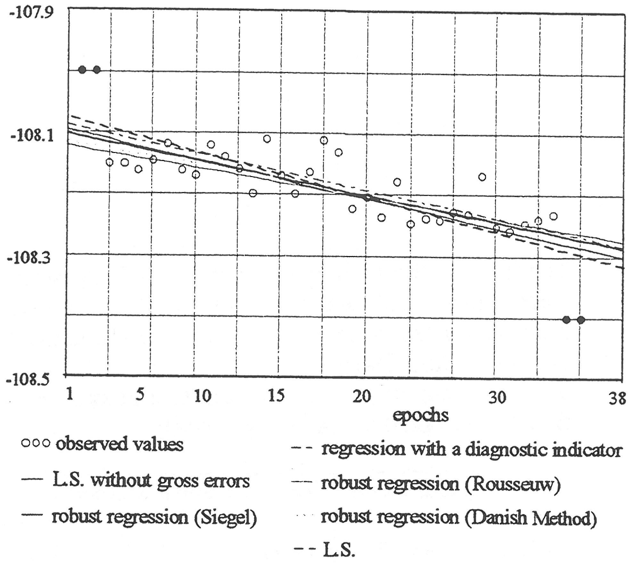


Fig. 7.4 Results of the adjustment

It can be seen from Fig.7.4 how all the robust straight lines are quite "concentrated" around the L.S. estimate in the absence of the four gross errors, and this is a proof of the reliability of robust estimates.

8. THE GPS MONITORING NETWORK FOR THE ASSISI LANDSLIDE

The robust estimate methods were next applied to a GPS monitoring network for the Assisi landslide.
The network is made up of six reference vertices placed in geologically stable areas, and fourteen control vertices.
The processing was done first on a series of four different GPS network simulations, and then on the reference point network of the Assisi landslide.

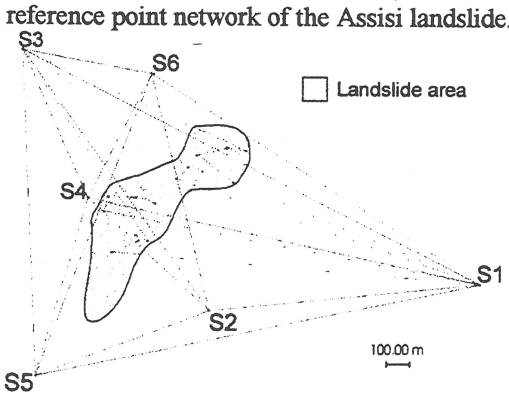


Fig. 8.1 The Assisi network

8.1 SIMULATIONS

The four different simulations differ from each other in the network geometry and the number of constraints, and therefore the redundancy.

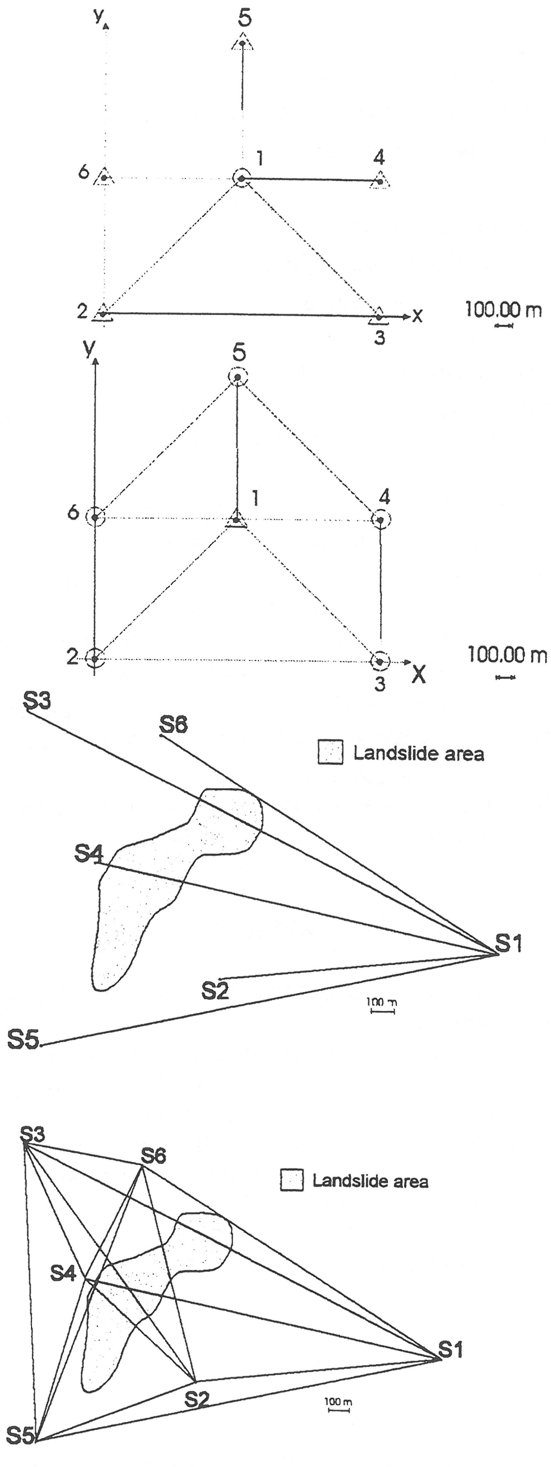


Fig. 8.1.1 The 4 simulations

In particular, the measurement vector was constructed by adding a random component with a mean of zero and a standard deviation of 5 mm to the "real" measurement vector.
The variance-covariance matrix was formed, imposing a 25 mm^2 variance and a covariance of 2.5 mm^2.
After a single base solution was carried out, four different situations were examined:

- a gross error of 20 cm in the 1st component of baseline 1-2;
- a gross error of 20 cm in the 2nd component of baseline 1-2;

- a gross error of 20 cm in the 3[st] component of baseline 1-2;
- a gross error of 20 cm in the norm of baseline1-2.

The following estimates were carried out:

- LS without gross errors;
- LS;
- Robust (Rousseeuw);
- robust (hyperbolic tangent);
- robust (Danish method);
- robust (Hampel).

From the results it was observed that:

- the LS solution was strongly influenced by gross errors, since it moves from the "real" parameter values;
- on the other hand, the robust solutions reduce their differences from these "real" values;
- Rousseeuw's estimator had the advantage of trending to "real" values (while the other robust estimators trend to the LS estimates without errors). However its convergence speed is not optimal;
- the Danish method presents the greatest convergence speed (5-6 iterations are sufficient);
- finally, the *regulation constants* of the various reweighing functions are function of the network redundancy.

8.2 THE ASSISI GPS REFERENCE NETWORK

Lastly, we analysed the monitoring network for the Assisi landslide.
Considering a minimum constraint datum (vertex 1 fixed and without errors) and a single base solution, a L.S. estimate was carried out, which was later subjected to a global test, to verify if the sample unit of weight variance came within its theoretic distribution.
WGS-84 coordinates of vertex 1

$$x_1 = \begin{pmatrix} 4554649.117 \\ 1021931.667 \\ 4333034.576 \end{pmatrix}$$

$\Delta_{i\,ran\,1}$	$\Delta_{i\,ran\,2}$	$\Delta_{i\,ran\,3}$	$\Delta_{i\,ran\,4}$
-173.958	1024.473	504.091	519.823
1527.801	307.687	-243.202	-2447.035
361.966	-1531.401	-550.101	-734.189
-784.27	-280.18	-105.652	1130.137
-521.157	-764.356	705.766	-398.08
1014.236	464.133	-144.937	-1386.448
-106.212	345.856	-678.615	626.036
-2292.156	-919.239	-1226.921	-154.879
102.172	-372.232	1159.172	-836.361
-398.439	-610.314	504.65	
-462.566	-2048.961	2754.719	
695.037	652.26	-797.21	

Tab. 8.2.1 Baseline measured

Global test :

$$\begin{vmatrix} \alpha = 5\% \\ r = 30 \end{vmatrix} \rightarrow \chi^2_{r,\alpha} = 43.77$$

$$T = \frac{\hat{\mathbf{v}}^T \cdot \mathbf{P} \cdot \hat{\mathbf{v}}}{\sigma_0^2} = \frac{r\,S_0^2}{\sigma_0^2} = 45.70$$

The test rejected the model. Therefore robust estimates were carried out according to the four estimators previously introduced (Rousseeuw, hyperbolic tangent, Danish method, Hampel).
The four estimates were subjected to a global test, and the tests accepted all the robust models.

	Rousseeuw	Tanh	Danish Method	Hampel
$\chi^2_{r,\alpha}$	43.77	43.77	43.77	43.77
T	13.94	12.02	21.41	29.82

For vertexes 2 and 6 in particular, the L.S. estimate moved away from the robust estimate. As this trend had been seen in previous controls, it leads to the validity of robust estimates.
In the figure below the adjusted coordinates of vertex 6 with the hyperbolic tangent estimator and with L.S. are shown.

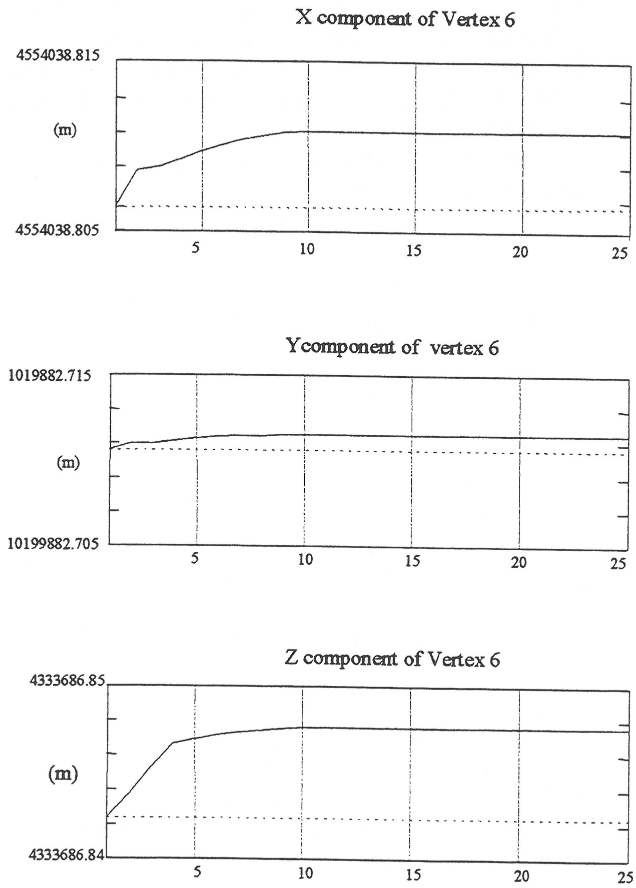


Fig. 8.2.1 Adjusted coordinates of vertex 6 with the hyperbolic tangent estimator and with L.S..

It can be noticed how the maximum movement between the two estimates is ≅ 5 mm, which is quite significant considering that the average movement observed is about 1 cm per year.

Vertex	Tanh-L.S. (mm)
X_6	4.48
Y_6	0.90
Z_6	5.23

Finally, on the histogram below a comparison between the results of the four estimators is given.

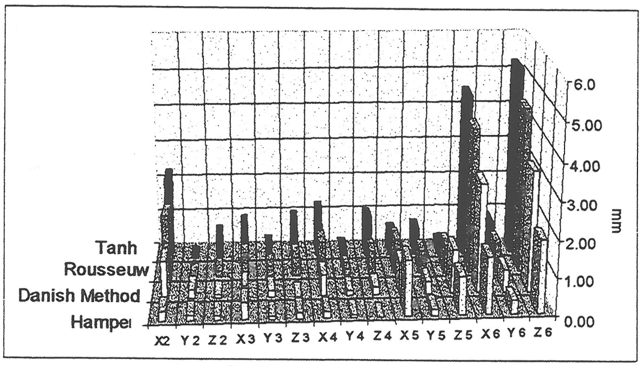


Fig. 8.2.2 Differences in absolute value between the 4 robust estimate and the L.S. estimate.

9. CONCLUSIONS

Although research is still in progress, it is nonetheless possible to draw some conclusions; fortunately we can say that robust estimates offer several advantages over the L. S. estimates:

- in the presence of gross errors they limit the estimate distortion;
- in the absence of contamination of the data the efficiency losses are also limited.

On the other hand some problems remain:

- apart for "Baarda data snooping" and Pope's τ test, which however come into the category of diagnostic indicators, few examples of robust procedures applied to geodetic problems exist.
- As the theory of robustness is relatively new, we haven not yet acquired the knowledge that would enable evaluation of the best estimators for the various different situations;
- Finally ,the computational calculation remains difficult.

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