ADJUSTMENT OF ALGEBRAIC SURFACES BY LEAST SQUARED DISTANCES

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ABSTRACT

GESTALTs are part of ORIENT, the hybrid adjustment program for photogrammetric, geodetic, and other observations. GESTALT surfaces are formalized as tri-variate polynomials. Further "form-parameters" allow convenient formulation of algebraic surfaces allowing the use of canonical forms (for e.g. ellipsoids or tori); nevertheless, these surfaces may have general position and attitude in space.
GESTALTs contain "fictitious" observations, such that points "lay on some surface" $\pm$ some accuracy.
The LLSQ-minimization principle for (implicit) functions is discussed: minimization of "algebraic error" versus minimization of "normal distance".
Derivatives for doing iterative LLSQ adjustment are presented supplemented by implementation aspects.

1 INTRODUCTION

The original requirements for designing ORIENT (Kager, 1976) as hybrid adjustment program for photogrammetric, geodetic, and other observations contained also the handling of points in planes (horizontal, vertical, general), and on straight lines (horizontal, vertical, general).
Moreover, the handling of street-surfaces for court expertises of traffic-accident scenes should be feasible. Curves in space seemed to be a further expansion of the concept allowing the orientation of (e.g.) photographs using line-features (tie-curves) additionally to (or instead of) tie-points. Investigating the algorithmics of the system, the question arose whether all these relations had to be implemented as separate features or if they could be handled in the frame of one common concept. The answer was the concept of GESTALTs introducing "fictitious" observations to photogrammetry. The applicability of the system was demonstrated in (Kager, 1980).
After the discovery that the algebraic formalism for GESTALTs also contained closed (=implicit) surfaces $f(x, y, z) = 0$, two problems appeared: first, the question of what to minimize when using an implicit equation, since it is homogeneus causing its residuals non-metric; second, the minor oddities that it is easy to adjust a general ellipsoid as a general quadric, but difficult to handle rotational ellipsoids in this way. The answers to these two problems shall be given in this paper: first, minimizing the squared distances from the surface; second, introducing "form-parameters" into the algorithm.

2 TRANSFORMATIONS AND THE FORMALISM OF GESTALTS

2.1 Spatial Similarity Transformation

The Spatial Similarity Transformation turns out as the foundations of all:

$$(\boldsymbol{x} - \boldsymbol{x}_o) = \lambda \cdot \boldsymbol{R}^T \cdot (\boldsymbol{X} - \boldsymbol{X}_o) =: \lambda \cdot \varrho \cdot \tilde{\boldsymbol{x}} \tag{1}$$

where:

$\boldsymbol{x} = (x, y, z)^T$	...	some observable point (e.g. in a photo, model, on a surface, etc.)
$\boldsymbol{x}_o = (x_o, y_o, z_o)^T$	...	the "internal reference point" (e.g. inner orientation of a photo)
λ	...	some scale factor (e.g. model scale)
$\boldsymbol{R}^T$	...	the rotation matrix transposed, $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{r})$
$\boldsymbol{r}$	...	the vector of rotation parameters (three angles or axis components)
$\boldsymbol{X} = (X, Y, Z)^T$	...	the object point
$\boldsymbol{X}_o = (X_o, Y_o, Z_o)^T$	...	the "external reference point" (e.g. projection center)
ϱ	...	some "normalization radius"
$\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}, \tilde{z})^T$	...	some local version of the normalized-rotated-reduced object point; a short-hand for:

$$\tilde{\boldsymbol{x}} = \frac{1}{\varrho} \cdot \boldsymbol{R}^T \cdot (\boldsymbol{X} - \boldsymbol{X}_o) = \tilde{\boldsymbol{x}}(\boldsymbol{X}, \boldsymbol{X}_o, \boldsymbol{r}) \tag{2}$$

It serves not only for Model Transformation but also as basics for Perspective Transformation as well as for Polar-point Transformation as well as for GESTALTs (i.e. surfaces) in ORIENT (ORIENT, 1995), (Kraus, 1996, B3.5).

2.2 The Concept of Fictitious Observations and Simple Gestalts

A fictitious observation is conceived as observation done in the mind - not using any physical measuring device - merely using experience and knowledge doing some geometric interpretation (Kager, 1976).
Anyway, a fictitious observation is defined as "distance of zero" between some point and some surface or curve. That is considered to be the only possible "measurement" to be done without using some physical instrument.
The first application of this idea is to use one (e.g. the third) row of (1) and "observe" this coordinate-component of $\boldsymbol{x}$ as zero-distance: $z = 0$. Formally this is done, simply multiplying (1) from the left with $\boldsymbol{e}_3^T$. Since $\boldsymbol{e}_3^T \boldsymbol{x}_o = z_o$, we have found the one "inner" parameter of this plane: the off-set of the local xy-plane (its attitude is defined by $\boldsymbol{R}$) - from the "external reference point" $\boldsymbol{X}_o$.
We can repeat this game with $\boldsymbol{e}_1^T$ and $\boldsymbol{e}_2^T$ as well; So we get two other planes: an yz-plane $x = 0$ and a zx-plane $y = 0$, any pair of them being orthogonal provided the same $\boldsymbol{R}$-matrix has been used.

The three types of Simple Gestalts are:

$$\begin{array}{lll} x\text{-Gestalt} & y\text{-Gestalt} & z\text{-Gestalt} \\ \boldsymbol{e}_1^T \boldsymbol{x} = x & \boldsymbol{e}_2^T \boldsymbol{x} = y & \boldsymbol{e}_3^T \boldsymbol{x} = z \\ \boldsymbol{e}_1^T \boldsymbol{x}_o = x_o & \boldsymbol{e}_2^T \boldsymbol{x}_o = y_o & \boldsymbol{e}_3^T \boldsymbol{x}_o = z_o \\ \boldsymbol{e}_1^T \tilde{\boldsymbol{x}} = \tilde{x} & \boldsymbol{e}_2^T \tilde{\boldsymbol{x}} = \tilde{y} & \boldsymbol{e}_3^T \tilde{\boldsymbol{x}} = \tilde{z} \end{array} \tag{3}$$

Another important feature is contained in that proposal: Parallelity of planes can be achieved by using one type of these planes with the same $\boldsymbol{R}$-matrix but different "inner" parameters $\boldsymbol{x}_o$ or different "external reference points" $\boldsymbol{X}_o$.

2.3 Gestalts with Curvature

It seemed a good idea to formulate (patches of) street-surfaces as bi-variate polynomials:

$$z_o = \sum_{i=0}^{9} \sum_{j=0}^{9} c_{ij} \cdot \tilde{x}^i \cdot \tilde{y}^j \tag{4}$$

the c_{ij} being coefficients for the $\tilde{x}^i \tilde{y}^j$-terms, the upper summing index "9" stemming from the coding scheme (see section 5.3). It can easily be seen that the above definition of a fictitious observation $\boldsymbol{e}_3^T \boldsymbol{x} = z = 0$ together with (4), interpreted as "inner" parameters $\boldsymbol{e}_3^T \boldsymbol{x}_o = z_o$, does not only the tilting of the planes but also the desired generalization from planes to surfaces with curvature. This being merely a question of the set of c_{ij} used which also determines the degree of the actual polynomial.
Since these c_{ij} "add" the higher degrees to the "simple GESTALTs" we call them "additional parameters" ADPAR. This is also an analogy to the introduction of additional parameters describing distortion and film shrinkage for photographs (Kraus, 1996, B3.5.6).
Pure formal generalization yields a GESTALT-surface as tri-variate polynomial:

$$z_o = \sum_{i=0}^{9} \sum_{j=0}^{9} \sum_{k=0}^{9} c_{ijk} \cdot \tilde{x}^i \cdot \tilde{y}^j \cdot \tilde{z}^k \tag{5}$$

Since this equation can also be written for x_o and y_o replacing c_{ijk} by a_{ijk} resp. b_{ijk} , we have a common scheme for any of them.
Some time after, it was discovered by chance that (5) contains also closed surfaces which was exploited to adjust circles and ellipses intersecting cylindric gestalts with planes in hybrid bundle adjustment (Kager, 1981).

Examples:

Set of coefficients c_{ijk} for a 3-axes ellipsoid in canonical form:
$\{c_{ijk}\} = \{c_{000}, c_{200}, c_{020}, c_{002}\}$
Set of coefficients c_{ijk} for a 3-axes ellipsoid with attitude:
$\{c_{ijk}\} = \{c_{000}, c_{200}, c_{110}, c_{020}, c_{101}, c_{011}, c_{002}\}$
Set of coefficients c_{ijk} for a 3-axes ellipsoid with attitude and shift:
$\{c_{ijk}\} = \{c_{000}, c_{100}, c_{010}, c_{001}, c_{200}, c_{110}, c_{020}, c_{101}, c_{011}, c_{002}\}$

2.4 Gestalts with Form-Parameters

Bearing in mind the inconvenience of interpreting the 2^{nd} order coefficients contained in a quadric, we desire a more direct formulation using the length of its axes resp. of the radii of circles as parameters.
To achieve this, a further generalization of the basic concept allows the usage of up to four form-parameters $\in \{q, r, s, t\}$ for any GESTALT-surface.

$$z_o = \sum_{i=0}^{9}\sum_{j=0}^{9}\sum_{k=0}^{9}\sum_{l=0}^{9}\sum_{m=0}^{9}\sum_{n=0}^{9}\sum_{o=0}^{9} c_{ijklmno} \cdot \tilde{x}^i \cdot \tilde{y}^j \cdot \tilde{z}^k \cdot q^l \cdot r^m \cdot s^n \cdot t^o \tag{6}$$

Surface	count	form-parameters	
Sphere	1	r	radius r
Ellipsoid	3	r, s, t	length of three semi-axes a, b, c
Rotational Ellipsoid	2	r, t	length of equator- and polar-semi-axis $a = b, c$
Torus	2	r, t	inner and outer radius r, R

Table 1: Surface description by form-parameters

Examples for sets of coefficients $c_{ijklmno}$ for some surfaces from table 1 (literally given values for some coefficients indicate them being constant, whereas the others are to be determined by adjustment; for coding scheme see section 5.3):

- A sphere $\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2 = r^2$ in canonical form using form-parameter r as radius:
  $\{c_{ijklmno}\} = \{c_{0000200} = -1, c_{2000000} = 1, c_{0200000} = 1, c_{0020000} = 1\}$
- A 3-axes ellipsoid $s^2 \cdot t^2 \cdot \tilde{x}^2 + r^2 \cdot t^2 \cdot \tilde{y}^2 + r^2 \cdot s^2 \cdot \tilde{z}^2 = r^2 \cdot s^2 \cdot t^2$ in canonical form using the length of the semi-axes (a, b, c) as form-parameters $r = a, s = b, t = c$:
  $\{c_{ijklmno}\} = \{c_{0000222} = -1, c_{2000022}, c_{0200202}, c_{0020220}\}$
- The same ellipsoid but using the reciprocal semi-axes as form-parameters $r = 1/a, s = 1/b, t = 1/c$
  $\{c_{ijklmno}\} = \{c_{0000000} = -1, c_{2000200}, c_{0200020}, c_{0020002}\}$
- A 2-axes (rotational) ellipsoid $r^2 \cdot t^2 \cdot \tilde{x}^2 + r^2 \cdot t^2 \cdot \tilde{y}^2 + r^4 \cdot \tilde{z}^2 = r^4 \cdot t^2$ using form-parameters $r = a = b, t = c$ for length of the semi-axes:
  $\{c_{ijklmno}\} = \{c_{0000402} = -1, c_{2000202}, c_{0200202}, c_{0020400}\}$

2.5 Gestalts in General Design

We prefer to describe GESTALT-surfaces with form-parameters in pure vector notation; later on, we will see that this will ease differentiation for deriving the linearized observation equations.

A 3D-vector of surfaces (corresponding to (1)) may be set-up by the list of ADPAR-vectors $\boldsymbol{a}_{ijklmno} := (a, b, c)^T_{ijklmno}$:

$$\begin{aligned} \boldsymbol{x}_o &= \sum_{i=0}^{9}\sum_{j=0}^{9}\sum_{k=0}^{9}\sum_{l=0}^{9}\sum_{m=0}^{9}\sum_{n=0}^{9}\sum_{o=0}^{9} \boldsymbol{a}_{ijklmno} \cdot (e_1^T\tilde{\boldsymbol{x}})^i \cdot (e_2^T\tilde{\boldsymbol{x}})^j \cdot (e_3^T\tilde{\boldsymbol{x}})^k \cdot (e_1^T\boldsymbol{q})^l \cdot (e_2^T\boldsymbol{q})^m \cdot (e_3^T\boldsymbol{q})^n \cdot (e_4^T\boldsymbol{q})^o \\ &= \boldsymbol{x}_o(\tilde{\boldsymbol{x}}, \boldsymbol{a}, \boldsymbol{q}) \end{aligned} \tag{7}$$

where $\boldsymbol{a} := \{\, \boldsymbol{a}_{ijklmno} \,\}$ is the set of relevant coefficient vectors, and $\boldsymbol{q} = (q, r, s, t)^T$ is the vector of form-parameters. This definition allows for substitution of $\boldsymbol{x}_o$ into (1) as function of the parameters $\tilde{\boldsymbol{x}}$, $\boldsymbol{a}$, and $\boldsymbol{q}$.
In return, it is possible to select one scalar surface-equation for the g^{th} component from this vector of surfaces (with $g \in \{1, 2, 3\}$ corresponding to a gestalt's x-, y-, or z-equation (3)):

$$(\boldsymbol{x}_o)_g = e_g^T \cdot \boldsymbol{x}_o = (\boldsymbol{x}_o(\tilde{\boldsymbol{x}}, \boldsymbol{a}, \boldsymbol{q}))_g = e_g^T \cdot \boldsymbol{x}_o(\tilde{\boldsymbol{x}}, \boldsymbol{a}, \boldsymbol{q}) =$$

$$\sum_{i=0}^{9}\sum_{j=0}^{9}\sum_{k=0}^{9}\sum_{l=0}^{9}\sum_{m=0}^{9}\sum_{n=0}^{9}\sum_{o=0}^{9} (e_g^T\boldsymbol{a}_{ijklmno}) \cdot (e_1^T\tilde{\boldsymbol{x}})^i \cdot (e_2^T\tilde{\boldsymbol{x}})^j \cdot (e_3^T\tilde{\boldsymbol{x}})^k \cdot (e_1^T\boldsymbol{q})^l \cdot (e_2^T\boldsymbol{q})^m \cdot (e_3^T\boldsymbol{q})^n \cdot (e_4^T\boldsymbol{q})^o \tag{8}$$

Choosing $g = 3$, (8) comes out to be identical to (6)

In the implementation, a GESTALT is such a vector of surfaces - see section 5.1.

3 MINIMIZATION BY LEAST SQUARED DISTANCES

At first we have to consider the observation vector $\boldsymbol{x}$ in (1);
$\boldsymbol{x}$ being a "true" quantity, it has to be split into two components: the observation $\overline{\boldsymbol{x}}$ and its correction $\boldsymbol{v}$:

$$\boldsymbol{x} = \overline{\boldsymbol{x}} + \boldsymbol{v} \tag{9}$$

An observed GESTALT point yields

$$\overline{\boldsymbol{x}} := 0 \tag{10}$$

due to the above definition of fictitious observations.

This and the original concept of GESTALTs according (1), based upon explicit functions describing the surface (4), yields:

$$z - z_o = \lambda \cdot \varrho \cdot \tilde{z} = v_z - z_o \qquad \text{therefore} \qquad v_z = z_o(\tilde{\boldsymbol{x}}, \boldsymbol{a}, \boldsymbol{q}) + \varrho \cdot \tilde{z} \tag{11}$$

since $\lambda := 1$, and $\varrho > 0$ being an arbitrary normalization radius.

From the LLSQ-point of view this implies that the correction v_z is measured along the local z-axis since v_z is directly related to $z = \varrho \cdot \tilde{z}$. This we might call "minimization by least squared coordinate discrepancies".

It is strict for planes, as given in (3), for Simple Gestalts using a rotation matrix for determining the tiltedness.

It is weak from a theoretic point of view for planes, the tilt of which is defined by utmost linear terms in (4) since the correction is not measured orthogonally to the plane.

It is weak but acceptable from a practical point of view for street-surfaces which are not much tilted, or, if "height-discrepancies" are even desired for minimization in (4).

But it is really inacceptible for somewhat bent or even closed surfaces.

Nevertheless, provided a plausible r.m.s. accuracy of the surface's observation, this approach can be easily homogenized together with photo-, model- and even polar-coordinates in hybrid adjustment (Kraus, 1996, B3.5.10).

Closed surfaces are best described using implicitly given formulae. Setting $\lambda := 0$ in (11), we arrive at

$$z - z_o = 0 = v_z - z_o \qquad \text{therefore} \qquad v_z = z_o = z_o(\tilde{\boldsymbol{x}}, \boldsymbol{a}, \boldsymbol{q}) \tag{12}$$

with $z_o(\tilde{\boldsymbol{x}}, \boldsymbol{a}, \boldsymbol{q})$ being the formal function (6) vectorized according (8) with GESTALT coefficients $\boldsymbol{a}$ and form parameters $\boldsymbol{q}$. The discrepancy v_z in (12) appears now as pure functional residual; moreover, as residual of an implicit equation since $z_o(\tilde{\boldsymbol{x}}, \boldsymbol{a}, \boldsymbol{q})$ shall represent some closed surface $f(\tilde{x}, \tilde{y}, \tilde{z}) = f(\tilde{\boldsymbol{x}}) = 0$. An implicit function may be multiplied with an arbitrary factor $\alpha \neq 0$ letting the geometric appearance of the surface unchanged because this proportionality factor α can be absorbed by the coefficients $\boldsymbol{a} \rightarrow \alpha * \boldsymbol{a}$ for instance. $z_o(\tilde{\boldsymbol{x}}, \alpha * \boldsymbol{a}, \boldsymbol{q}) = \alpha * z_o(\tilde{\boldsymbol{x}}, \boldsymbol{a}, \boldsymbol{q})$ yields v_z versus $\alpha * v_z =: w$ as discrepancy to be LLSQ'd also known as "minimization of algebraic error". This property does not matter for a point exactly on the surface since in this case $v_z = 0$ holds true; in hybrid adjustment on the other hand - when that point is overdetermined by e.g. rays from photos or it is also measured by polar coordinates - we might bias our results neglecting homogenization of weights (see (Kraus, 1996, B3.5.6)). So, the right choice of α is responsible for what will be minimized in LLSQ adjustment. One should also confer to those examples of an ellipsoid with semi-axes (a, b, c) versus an ellipsoid with reciprocal semi-axes $(1/a, 1/b, 1/c)$ as form-parameters near the end of section 2.4; $\alpha = r^2 \cdot s^2 \cdot t^2$ illustrates the phenomemon of implicit functions (not only in the case of canonic quadrics).

Some people don't consider that important enough and do adjustment of a circle with the observation equation $v = x^2 + y^2 - r^2$ not worrying about minimizing corrections of r^2 instead of corrections of r by LLSQ.

For some time, we used an approximate method to alleviate the mis-homogenization: simply doing some special weighting modulating the standard weighting due to accuracy. E.g. adjusting a circle, it is simple to multiply the observation equation $w = x^2 + y^2 - r^2$ with $1/r_o$, this r_o being an approximation of r yielding $v \doteq w/r_o = x^2/r_o + y^2/r_o - r^2/r_o$. With this simple measure, circles and also (more or less circular) quadrics may be adjusted with "near-metric" corrections - eased by the fact that "weighting" an observation equation is also simply multiplying it with a factor.

This sloppyness being not satisfying, we will try to get a metrical measure for the discrepancy of a point $\boldsymbol{X}$ resp. $\tilde{\boldsymbol{x}}$ used as argument for an implicit function. We consider the euklidean distance measured orthogonally to the surface appropriate to serve as such metrical measure.

3.1 Metrification of Algebraic Error

We assume the surface given by an implicit function $\varphi(x, y, z) = \varphi(\boldsymbol{x}) = 0$. Figure 1 shows this situation.

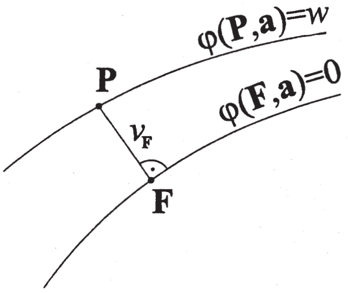

Figure 1: Potential surfaces - Normal in footing point $\boldsymbol{F}$.

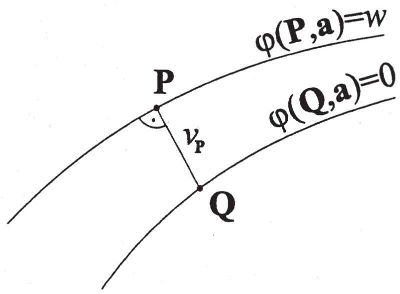

Figure 2: Potential surfaces - Normal in object point $\boldsymbol{P}$.

$\varphi(\boldsymbol{F}, \boldsymbol{a}) = 0$ resembles the true (resp. adjusted) surface with the footing point $\boldsymbol{F}$ for the adjusted point $\boldsymbol{P}$. $\boldsymbol{P}$ itself is used as argument for the surface φ and yields a contradiction $\varphi(\boldsymbol{P}, \boldsymbol{a}) = w$. This can be interpreted as if space is filled by a scalar field (like a potential field) and the desired surface represents potential *zero*; the final point $\boldsymbol{P}$ then will/might also lay on an iso-surface - but of potential w.

Bearing in mind to measure the correction v along the surface's normal in $\boldsymbol{F}$

$$\boldsymbol{n}_F := \boldsymbol{n}(\boldsymbol{F}, \boldsymbol{a}) = \left(\frac{\partial\varphi(\boldsymbol{F}, \boldsymbol{a})}{\partial \boldsymbol{F}}\right)^T = \nabla\varphi(\boldsymbol{F}, \boldsymbol{a}) \tag{13}$$

we use the gradient of the surface φ :

$$\boldsymbol{P} = \boldsymbol{F} + \boldsymbol{v}_F = \boldsymbol{F} + \mu_F \cdot \boldsymbol{n}_F \qquad \text{therefore} \qquad v_F = \mu_F \cdot \sqrt{\boldsymbol{n}_F^T \boldsymbol{n}_F} \tag{14}$$

with μ_F being a proportionality factor for scaling $\boldsymbol{n}_F$ and $v_F^2 := \boldsymbol{v}_F^T \boldsymbol{v}_F$.
We can describe w by Taylor-Series expansion of the zero-potential:

$$\begin{aligned} w = \varphi(\boldsymbol{P}, \boldsymbol{a}) = \varphi(\boldsymbol{F} + \mu_F \cdot \boldsymbol{n}_F, \boldsymbol{a}) &= \varphi(\boldsymbol{F}, \boldsymbol{a}) + \frac{\partial\varphi(\boldsymbol{F}, \boldsymbol{a})}{\partial \boldsymbol{F}} \cdot \mu_F \cdot \boldsymbol{n}_F + \cdots \text{higher order neglected} \\ &\doteq \varphi(\boldsymbol{F}, \boldsymbol{a}) + \mu_F \cdot \boldsymbol{n}_F^T \boldsymbol{n}_F = \mu_F \cdot \boldsymbol{n}_F^T \boldsymbol{n}_F \end{aligned} \tag{15}$$

using (13) and since $\varphi(\boldsymbol{F}, \boldsymbol{a}) = 0$ yielding

$$\mu_F \doteq \frac{w}{\boldsymbol{n}_F^T \boldsymbol{n}_F} \qquad \text{with (14) therefore} \qquad v_F = \frac{w}{\sqrt{\boldsymbol{n}_F^T \boldsymbol{n}_F}} \tag{16}$$

For practical computation we have $\boldsymbol{F}$ not available at first. We could iterate that point $\boldsymbol{F}$ by some steepest-descent method as done in (Forkert, 1994) for ORIENT's 3D-splines, but here we want to circumvent that effort.

Assuming a "best corresponding" point Q on the final surface $\varphi(\boldsymbol{Q}, \boldsymbol{a}) = 0$, we get analogously due to figure 2:

$$\boldsymbol{n}_P := \boldsymbol{n}(\boldsymbol{P}, \boldsymbol{a}) = \left(\frac{\partial\varphi(\boldsymbol{P}, \boldsymbol{a})}{\partial \boldsymbol{P}}\right)^T = \nabla\varphi(\boldsymbol{P}, \boldsymbol{a}) \tag{17}$$

we use the gradient of the surface φ :

$$\boldsymbol{Q} = \boldsymbol{P} + \boldsymbol{v}_P = \boldsymbol{P} + \mu_P \cdot \boldsymbol{n}_P \qquad \text{therefore} \qquad v_P = \mu_P \cdot \sqrt{\boldsymbol{n}_P^T \boldsymbol{n}_P} \tag{18}$$

with μ_P being a proportionality factor for scaling $\boldsymbol{n}_P$ and $v_P^2 := \boldsymbol{v}_P^T \boldsymbol{v}_P$.
We can now describe the zero-potential by Taylor-Series expansion from potential w :

$$\begin{aligned} 0 = \varphi(\boldsymbol{Q}, \boldsymbol{a}) = \varphi(\boldsymbol{P} + \mu_P \cdot \boldsymbol{n}_P, \boldsymbol{a}) &= \varphi(\boldsymbol{P}, \boldsymbol{a}) + \frac{\partial\varphi(\boldsymbol{P}, \boldsymbol{a})}{\partial \boldsymbol{P}} \cdot \mu_P \cdot \boldsymbol{n}_P + \cdots \text{higher order neglected} \\ &\doteq \varphi(\boldsymbol{P}, \boldsymbol{a}) + \mu_P \cdot \boldsymbol{n}_P^T \boldsymbol{n}_P = \mu_P \cdot \boldsymbol{n}_P^T \boldsymbol{n}_P \end{aligned} \tag{19}$$

using (17) and since $\varphi(\boldsymbol{P}, \boldsymbol{a}) = w$ yielding

$$\mu_P \doteq \frac{w}{n_P^T n_P} \quad \text{with (18) therefore} \quad v_P = \frac{w}{\sqrt{n_P^T n_P}} \tag{20}$$

So, we might accept this short-cut using

$$v \; := \; v_P \; = \; \frac{\varphi(\boldsymbol{P}, \boldsymbol{a})}{||\nabla\varphi(\boldsymbol{P}, \boldsymbol{a})||} \; = \; \frac{w}{||n_P||} \tag{21}$$

instead of v_F since these two normals should not differ too much in length. For measuring an observation discrepancy in LLSQ, that's certainly fine; for convergency behaviour, problems may be expected; moreover since doing LLSQ adjustment iteratively, the surface as well as the object point P is only approximately available; needing investigation in the future.

4 DERIVATIVES

4.1 Derivatives for Spatial Similarity Transformation

At first we have to cite the differentials for the of Spatial Similarity Transformation to be able to set up the linearized observation equations. Equation (1) contains the variates $\boldsymbol{x}_o$, λ , $\boldsymbol{r}$, $\boldsymbol{X}$, and $\boldsymbol{X}_o$. Differentiation of the observable $\boldsymbol{x}$ in (1) by these variates yields the already well known differential quotients below. For convenience, we use (1) as follows:

$$\boldsymbol{x} \; = \; \boldsymbol{x}_o \; + \; \lambda \cdot \varrho \cdot \tilde{\boldsymbol{x}}(\boldsymbol{X}, \boldsymbol{X}_o, \boldsymbol{r}) \tag{22}$$

Derivatives:

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}_o} \; = \; \boldsymbol{I} \tag{23}$$

$$\frac{\partial \boldsymbol{x}}{\partial \lambda} \; = \; \varrho \cdot \tilde{\boldsymbol{x}} \; = \; \boldsymbol{R}^T \cdot (\, \boldsymbol{X} \; - \; \boldsymbol{X}_o \,) \tag{24}$$

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \; = \; +\lambda \cdot \varrho \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{X}} \; = \; +\lambda \cdot \boldsymbol{R}^T \tag{25}$$

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}_o} \; = \; -\lambda \cdot \varrho \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{X}_o} \; = \; -\lambda \cdot \boldsymbol{R}^T \; = \; -\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \tag{26}$$

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}} \; = \; \lambda \cdot \varrho \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{r}} = \lambda \cdot \frac{\partial(\boldsymbol{R}^T \cdot (\boldsymbol{X} - \boldsymbol{X}_o))}{\partial \boldsymbol{r}} \tag{27}$$

With $\boldsymbol{I}$ being the Identity matrix; the latter fraction $\frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{r}}$ being also a $(3,3)$-matrix the derivation of which shall not be exploited to more detail in this context (see also (Dermanis, 1994)).

4.2 Derivatives for Gestalts

Since in the case of GESTALTS the $\boldsymbol{x}_o = \boldsymbol{x}_o(\tilde{\boldsymbol{x}}, \boldsymbol{a}, \boldsymbol{q})$ are themselves functions of other variates (see (2)), we have - after substitution to (1) - to differentiate (7) by $\tilde{\boldsymbol{x}}$, $\boldsymbol{a}$, and $\boldsymbol{q}$ applying the chain-rule esp. for $\boldsymbol{X}$, $\boldsymbol{X}_o$, and $\boldsymbol{r}$. We are taking into account that for all GESTALTs $\lambda = const$, and $\varrho = const$, and especially for implicitly given GESTALTs $\lambda = 0$ and also (10); instead of $\boldsymbol{v}$ as in (9) we use $\boldsymbol{w}$ to indicate eventually algebraic error as discussed in section 3.1:

$$\boldsymbol{x} \; = \; \boldsymbol{x}_o(\tilde{\boldsymbol{x}}(\boldsymbol{X}, \boldsymbol{X}_o, \boldsymbol{r}), \boldsymbol{a}, \boldsymbol{q}) \; + \; \lambda \cdot \varrho \cdot \tilde{\boldsymbol{x}}(\boldsymbol{X}, \boldsymbol{X}_o, \boldsymbol{r}) \; = \; \bar{0} \; + \; \boldsymbol{w} \tag{28}$$

Derivatives (using the definition $\tilde{\boldsymbol{N}}^T \; := \; (\frac{\partial \boldsymbol{x}_o}{\partial \tilde{\boldsymbol{x}}} + \lambda \cdot \varrho \cdot \boldsymbol{I})$):

$$\frac{\partial \boldsymbol{x}}{\partial \lambda} \; = \; 0 \tag{29}$$

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \; = \; \frac{\partial \boldsymbol{x}_o}{\partial \tilde{\boldsymbol{x}}} \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{X}} + \lambda \cdot \varrho \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{X}} \; = \; (\frac{1}{\varrho} \cdot \frac{\partial \boldsymbol{x}_o}{\partial \tilde{\boldsymbol{x}}} + \lambda \cdot \boldsymbol{I}) \cdot \boldsymbol{R}^T \; = \; \frac{1}{\varrho} \cdot \tilde{\boldsymbol{N}}^T \cdot \boldsymbol{R}^T \; =: \; \boldsymbol{N}^T \tag{30}$$

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}_o} \; = \; \frac{\partial \boldsymbol{x}_o}{\partial \tilde{\boldsymbol{x}}} \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{X}_o} - \lambda \cdot \varrho \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{X}_o} \; = \; (\frac{-1}{\varrho} \cdot \frac{\partial \boldsymbol{x}_o}{\partial \tilde{\boldsymbol{x}}} - \lambda \cdot \boldsymbol{I}) \cdot \boldsymbol{R}^T \; = \; -\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \; = \; -\boldsymbol{N}^T \tag{31}$$

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}} \; = \; \frac{\partial \boldsymbol{x}_o}{\partial \tilde{\boldsymbol{x}}} \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{r}} + \lambda \cdot \varrho \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{r}} \; = \; (\frac{1}{\varrho} \cdot \frac{\partial \boldsymbol{x}_o}{\partial \tilde{\boldsymbol{x}}} + \lambda \cdot \boldsymbol{I}) \cdot \frac{\partial(\boldsymbol{R}^T \cdot (\boldsymbol{X} - \boldsymbol{X}_o))}{\partial \boldsymbol{r}} \; = \; \tilde{\boldsymbol{N}}^T \cdot \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{r}} \tag{32}$$

$$\frac{\partial x_o}{\partial a_{ijklmno}} = (e_1^T\tilde{x})^i \cdot (e_2^T\tilde{x})^j \cdot (e_3^T\tilde{x})^k \cdot (e_1^T q)^l \cdot (e_2^T q)^m \cdot (e_3^T q)^n \cdot (e_4^T q)^o \cdot I \tag{33}$$

$$\frac{\partial x_o}{\partial q} = \sum_{i=0}^{9}\sum_{j=0}^{9}\sum_{k=0}^{9}\sum_{l=0}^{9}\sum_{m=0}^{9}\sum_{n=0}^{9}\sum_{o=0}^{9} a_{ijklmno} \cdot (e_1^T\tilde{x})^i \cdot (e_2^T\tilde{x})^j \cdot (e_3^T\tilde{x})^k \tag{34}$$

$$[\ +l \cdot (e_1^T q)^{l-1} \cdot (e_2^T q)^m \cdot (e_3^T q)^n \cdot (e_4^T q)^o \cdot e_1^T + m \cdot (e_1^T q)^l \cdot (e_2^T q)^{m-1} \cdot (e_3^T q)^n \cdot (e_4^T q)^o \cdot e_2^T +$$
$$+n \cdot (e_1^T q)^l \cdot (e_2^T q)^m \cdot (e_3^T q)^{n-1} \cdot (e_4^T q)^o \cdot e_3^T + o \cdot (e_1^T q)^l \cdot (e_2^T q)^m \cdot (e_3^T q)^n \cdot (e_4^T q)^{o-1} \cdot e_4^T\ ]$$

with $\tilde{N}$ resp. N containing as columns the normals of the three potential surfaces in the local resp. global system, and:

$$\frac{\partial x_o}{\partial \tilde{x}} = \sum_{i=0}^{9}\sum_{j=0}^{9}\sum_{k=0}^{9}\sum_{l=0}^{9}\sum_{m=0}^{9}\sum_{n=0}^{9}\sum_{o=0}^{9} a_{ijklmno} \cdot (e_1^T q)^l \cdot (e_2^T q)^m \cdot (e_3^T q)^n \cdot (e_4^T q)^o \cdot \tag{35}$$

$$[\ i \cdot (e_1^T\tilde{x})^{i-1} \cdot (e_2^T\tilde{x})^j \cdot (e_3^T\tilde{x})^k \cdot e_1^T + j \cdot (e_1^T\tilde{x})^i \cdot (e_2^T\tilde{x})^{j-1} \cdot (e_3^T\tilde{x})^k \cdot e_2^T + k \cdot (e_1^T\tilde{x})^i \cdot (e_2^T\tilde{x})^j \cdot (e_3^T\tilde{x})^{k-1} \cdot e_3^T\ ]$$

Notice well the dyadic products $a_{ijklmno} \cdot e_d^T$ with $d \in \{1,2,3,4\}$ in above derivatives!

4.3 Derivatives for Normalized Gestalts

To make things easier, we handle only implicit GESTALTs now: $\lambda = 0$. So, (28) becomes in accordance with (21):

$$x_n := \frac{x}{||n_p||} = \frac{x_o(\tilde{x}, a, q)}{||n_p||} = \frac{x_o(\tilde{x}, a, q)}{||n_p(\tilde{x}, a, q)||} = \bar{0} + v \tag{36}$$

Since the denominator is also a function, the derivatives become more complicated. Additionally to section 4.2, the quotient-rule has to be taken into account to get the various $\frac{\partial x_n}{\partial \cdot}$. Due to the space-limitation of this paper, the reader is invited to try it himself. He should then explore further the important role of the normal vectors contained in $\tilde{N}$ resp. $N$and he might come finally to the derivatives of these normals: the Hessian Matrix being the 2^{nd} derivative of the surface.

5 PROGRAMMING ASPECTS

5.1 Implementation

A GESTALT is a vector of surfaces (7) but not all of its components are validated (i.e. turned on). If one component is validated, this GESTALT represents a surface; if two components are validated, this GESTALT represents a curve. The ADPAR representing $e_g^T a := \{\ e_g^T a_{ijklmno}\ \}$ build a list of "1D-points", the identifier of every one contains g and the actual index for $a_{ijklmno}$ (8) (see section 5.3). A "status" controls (besides others) whether that coefficient is treated as fixed (=constant) parameter or as free unknown to be determinable in adjustment.

5.2 Numerics

Since polynomials are known as numerically critical under certain circumstances, two measures were taken to ensure numerical stability for the polynomial expressions in e.g. (7). Numerical mathematics suggests the domain of polynomials being the interval $[-1, +1]$ to grant stability. The definition of the point-argument $\tilde{x}$ (2) used for all polynomial expressions does as well the centering around *zero* as the normalization to that *unit*-interval.
The centering, done by the choice of the "external reference point" X_o is responsible mainly for avoidance of numeric extinction, whereas the "normalization radius" ϱ keeps the values of the coefficients in a moderate range avoiding exponent overflows. It has to be stressed that the definition of higher degree/order GESTALTs requires some skillness of the operator: to define a suitable reference point in the center of the area of interest and to estimate the radius of that area. GESTALTs defined in canonical form may be considered stable, usually.

5.3 Coding Scheme

Every required individual ADPAR coefficient $e_g^T a_{ijklmno}$ has an identifier; it is coded as "point number":

identifier	g	i	j	k	l	m	n	o	$\rightarrow$ type g of equation and exponents $i \cdots o$
e.g.	3	0	2	1	0	2	0	0	$\Rightarrow$ $z_o = \cdots + c_{0210200} \cdot \tilde{y}^2 \tilde{z} r^2 + \cdots$ in scalar notation like (6)
base		x	y	z	q	r	s	t	
form par.	g	9	9	9	0	0	0	4	$\rightarrow q$
form par.	g	9	9	9	0	0	0	3	$\rightarrow r$
form par.	g	9	9	9	0	0	0	2	$\rightarrow s$
form par.	g	9	9	9	0	0	0	1	$\rightarrow t$

5.4 Example

This example shows the definition of a torus using the ORIENT command language DIRAN (comments - to end of line - are indicated with $c):

```
CRE   Gest 301 data(.dat) free end(-99) z=obs(
$c gijkqrst                 x^i  y^j  z^k S^s  T^t canonical form; i.e. in local coord-sys
  -34000000  1.   0.1 $c  x^4
  -30400000  1.   0.1 $c       y^4
  -30040000  1.   0.1 $c            z^4

  -32200000  2.   0.1 $c  x^2  y^2
  -32020000  2.   0.1 $c  x^2       z^2
  -30220000  2.   0.1 $c       y^2  z^2

  -32000020 -1.   0.1 $c  x^2            S^2
  -32000002 -1.   0.1 $c  x^2                 T^2
  -30200020 -1.   0.1 $c       y^2       S^2
  -30200002 -1.   0.1 $c       y^2            T^2
  -30020011 +2.   0.1 $c            z^2  S    T
  -30000022  1.   0.1 $c                 S^2  T^2

   39990002  1.  10.1 $c                 S        := (R+r)/rho0
   39990001  0.3 10.1 $c                      T   := (R-r)/rho0

      G1              $c subtype 1
   )  erp rotpar=obs=omfika(3 4 5 0 0 0.001) norm(5) take(G3) sig(.01).
```

The details shall not bother us here - the reader may try to verify the coefficients above with a suitable torus-formula from a text-book; nevertheless, one thing is remarkable and crucial with our definition of GESTALTs: since a torus is rotationally invariant (this one with respect to the $\tilde{z}$-axis) we have to take care of a singularity which will show up in adjustment; this singularity is stabilized, observing fictitionally that rotational parameter which is responsible for the z-rotation ($\kappa = 5^g \pm 0.001^g$).

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