A COMPACT VECTOR-SPACE ALOGARITHM FOR AN ANALYTICAL REDUCTION OF A STEREOGRAM

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ABSTRACT

The need for variety and better accuracy of the map products used in the rapidly evolving GIS technology has necessitated the use of analytical techniques in photogrammetric data processing. The ready availability of small but powerful computers which are able to support the computational requirements of rigorous solutions, and the flexibility to use analogue or digital input imageries in the mapping process have combine to make analytical procedures a routine utility. Quite often in all applications, the basic problem involves the extraction of spatial entities from images, and almost invariably, the most accurate mapping is achieved through rigorous treatment of stereo images. The software modules used in the operations are developed based on appropriate mathematical representations of the relationships between measurements of image features and the corresponding spatial objects.

From an abstract geometric consideration, each image of a stereogram may be visualized as a 3-D vector space whose elements are composed action, and the calibrated principal constant of the camera or sensor. The object space, in most practical cases, may also be represented simply as a 3-D vector space. When suitable coordinate systems are attached to these spaces, they become Euclidean spaces in which points may be represented simply in position vectors. This space conceptualization enables the use of vector symbology and linear algebra to develop compact computational algorithms for the reduction process.

Although the theoretical bases of the mathematical formulations used in the treatment of a stereogram are well known and in fact vector notations have been used to present them. Nevertheless, the computational schemes often adopted are based on long hand approach in which symbols are used to represent single variables, involving tedious algebraic manipulation. This inevitably leads to complicate computational procedures, devoid of clear geometric meaning and insightful appeal. This paper applies the ARDOVS concept, an analytical tool, to the solution of the stereogram problem. It is shown how this methodology provides a compact and consistent solution scheme which is easy to understand.

1. INTRODUCTION

The need for variety and better accuracy of the map products used in the rapidly evolving GIS technology has necessitated the use of analytical techniques in photogrammetric data processing. The ready availability of small but powerful computers which are able to support the computational requirements of rigorous solutions, and the flexibility to use analogue or digital input imageries in the mapping process have combined to make analytical procedures a routine utility. Quite often in all applications, the basic problem involves the extraction of spatial entities from images, and almost invariably, the most accurate mapping is achieved through rigorous treatment of stereo images. The software modules used in the operations are developed based on appropriate mathematical representations of the relationships between measurements of image features and the corresponding spatial objects.

From an abstract geometric consideration, each image of a stereogram may be visualized as a 3-D vector space (Figure 1) whose elements are composed action, and the calibrated principal constant of the camera or sensor. The object space, in most practical cases, may also be represented simply as a 3-D vector space. When suitable coordinate systems are attached to these spaces, they become Euclidean spaces in which points may be represented simply in

position vectors (figure 1). This vector space conceptualization enables the use of vector symbology and linear algebra to develop compact computational algorithms for the reduction process.

Although the theoretical bases of the mathematical formulations used in the treatment of a stereogram are well known and in fact vector notations have been used to present them, nevertheless, the computational schemes often adopted are based on long hand approach in which symbols are used to represent single variables, involving tedious algebraic manipulation. This inevitably leads to complicated computational procedures, a devoid of clear geometric meaning and insightful appeal. This paper applies the **ARDOVS** concept [Olaleye 1992], an analytical tool, to the solution of the stereogram problem. It is shown how this methodology provides a compact and consistent solution scheme which is easy to understand and general application.
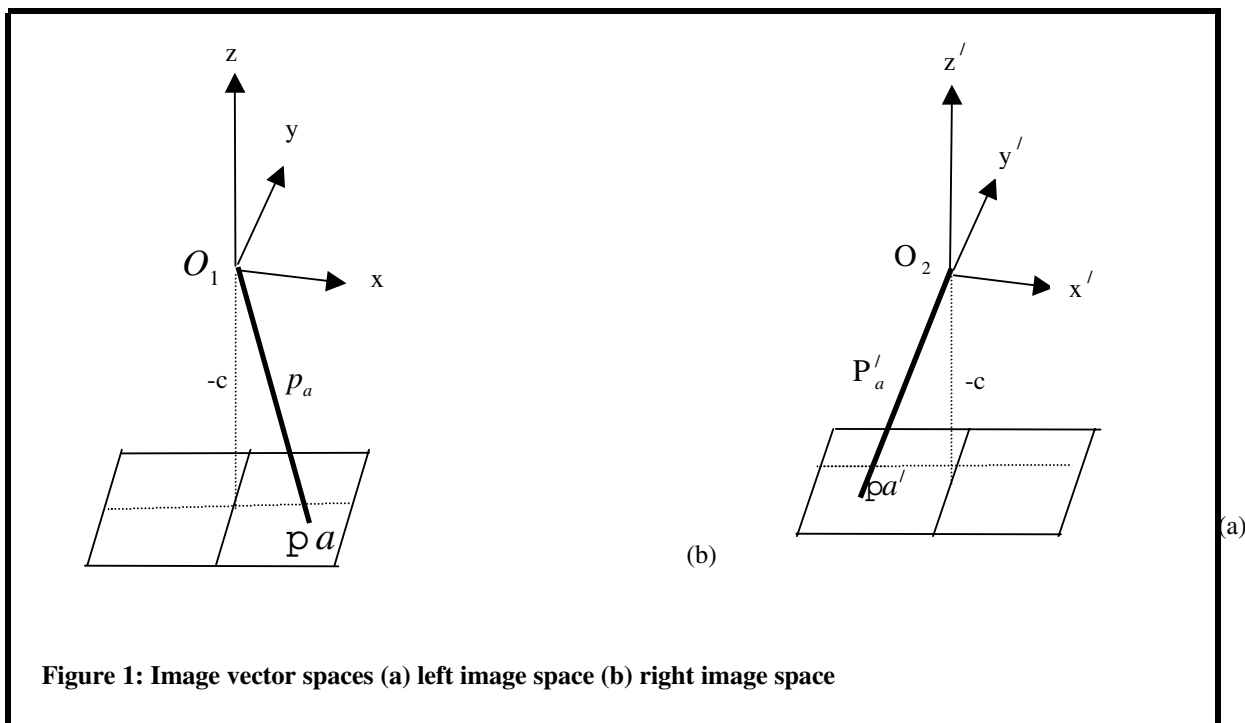

Figure 1: Image vector spaces (a) left image space (b) right image space

To begin with, the term "**ARDOVS**" is an acronym for "Apparent and Real Directions of Vector Spaces" and states that to every photogrammetric vector space, there is a set of natural directions (as seen by elements within space) and a set of apparent directions (as seen by elements outside the space. When two such spaces are to exchange elements, one space is always the fixed space, which in the ARDOVS concept is called the **R** - space with natural directions **i,j,k**, and apparent directions $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$, (figure 2a). The other space is then the movable space with natural directions **i',j',k'**, and apparent direction $\mathbf{C}_1, \mathbf{C}_2$, and $\mathbf{C}_3$ (Figure 2b).
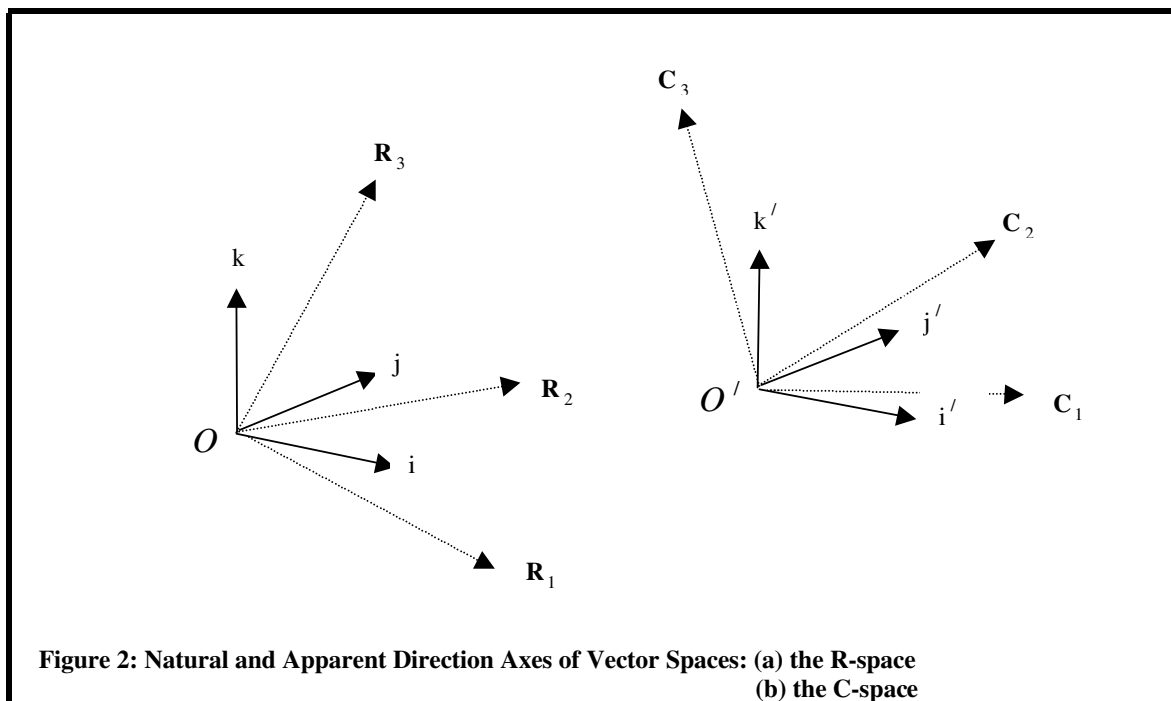
Figure 2: Natural and Apparent Direction Axes of Vector Spaces: (a) the R-space
(b) the C-space

The apparent direction vectors constitute the row- and column spaces of a 3-D rotation matrix and are expressed analytically as (Olaleye 1992):

R- Space direction vectors:

$$R_1 = \begin{bmatrix} \cos f \cos k \\ \sin w \sin f \cos k + \cos w \sin k \\ -\cos w \sin f \cos k + \sin w \sin k \end{bmatrix}^T$$

$$R_2 = \begin{bmatrix} -\cos f \sin k \\ -\sin w \sin f \sin k + \cos w \cos k \\ \cos w \sin f \sin k + \sin w \cos k \end{bmatrix}^T$$

$$R_3 = \begin{bmatrix} \sin f \\ -\sin w \cos w \\ \cos w \cos f \end{bmatrix}^T$$

C-space direction vectors

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$$C_1 = \begin{bmatrix} \cos f \cos k \\ -\cos f \sin k \\ \sin f \end{bmatrix} , \quad C_2 = \begin{bmatrix} \sin w \sin f \cos k + \cos w \sin k \\ -\sin w \sin f \sin k + \cos w \cos k \\ -\sin w \cos f \end{bmatrix} ,$$

$$C_3 = \begin{bmatrix} -\cos w \sin f \cos k + \sin w \sin k \\ \cos w \sin f \sin k + \sin w \cos k \\ \cos w \cos f \end{bmatrix} \tag{1}$$

Where w, f, k are the positive angular rotations about the i,j,k, axes that will bring the apparent axes of the movable space into the same orientation as the fixed space.

From an applications viewpoint, the ARDOVS concept serves a dual purpose. First, it may be used to simplify the derivation of formulations needed to transform elements between two vectors spaces if the apparent directions and other parameters are given: and secondly it may be used to derive computational algorithms to determine the apparent direction vectors and other parameters for use in subsequent applications. In this paper, we explore the first application of the concept, and for this, we make the assumption that the exterior orientation parameters (R.O + A.O) of the stereogram are already computed and that the imaging sensor has been properly calibrated. For our purpose, we shall use the exterior orientation data in two stages. First, we use the relative orientation information (R.O) to evaluate eqns 1 & 2 for the apparent direction vectors needed for the image space operations. Secondly, the absolute orientation information (A.O) will be used to evaluate eqns. 1&2 are evaluated with the R.O. elements, they refer to image space apparent directions and do not carry a superscript. When evaluated with the A.O. elements, they refer to object space directions and carry the star symbol (*) as a superscript.

Yet, the successful use of the **ARDOVS** techniques depends on correct identification of the **R**-space and the **C**-space in any problem situation. For the stereogram problem, this is easily achieved by recognizing that two image spaces (i.e. the left image space, the right image space) and an object space are involved, and by logical reasoning, the fixed space is the object space while the image spaces are the movable spaces. Therefore, for operations involving the image space and the object space, the image space is the **C**-space while the object space is the **R**-space. Moreover, the stereogram reduction process adopted in this paper involves transfer of elements between two image spaces which, according to the **ARDOVS** terminology, are moving and therefore are both **C**-spaces. Incidentally, the **ARDOVS** idea does not apply to two **C**-spaces, hence we must regard one of the image spaces as fixed and the other moving. The choice of one or the other space is purely a matter of preference, just similar to the usual practice in dependent pair relative orientation in which any of the images may be chosen as fixed while the other is moved. Consequently in this paper, the three vector spaces concerned are identified as follows:

(a)   for the image space operations:- the right image is a movable or **C**-space while the left image is fixed or **R**-space (see figure 4a & b)

(b)   for the object space operations:- the left image space is the **C** space while the object space is the **R**-space (see Figure 3c & figure 4)

This convention implies that the left image space will have its axes re-assigned when changing from the image space operation (see Figures 3a & b) to the object space operation (see Figure 3). Therefore, the left image space will be alternately referred to as the image **R**-space to object **C**- space is effected through a re-labelling of the left image axes. However, as said earlier, the apparent directions needed for all these operations are obtained by evaluating eqns. 1&2 with R.O and A.O elements respectively.

Having identified the spaces involved in the reduction of a stereogram, the application of the **ARDOVS** technique is premised on three operational rules (Olaleye 1992).

1.	An element of one space can only cross to the other space by projection onto the apparent directions of the new space. If the scales of the two spaces are different, the projection must be scaled to conform to the new space.

2.	Any element moving from the **C**-space to the **R**-space must add to itself after projection, the vector which located the origin of the **C**-space.

3.	Any element moving from the **R**-space to the **C**- space must reduce itself before projection, by the vector which locates the origin of the **C**-space.

	These rules, aided by simple vector algebra produce the reduction equations. As will be demonstrated with an example, the **ARDOV's** development leads to routine simplicity of the reduction process. However, a key factor to bear in mind in the present problem is that the image **R**-space and the object **C**-space are one and the same space except that depending on the operation being performed, its axes are labelled with the appropriate direction vectors. Also, when the axes direction vectors carry the superscript (*) notation, they refer to an object space operation, otherwise, an image space operation is implied.
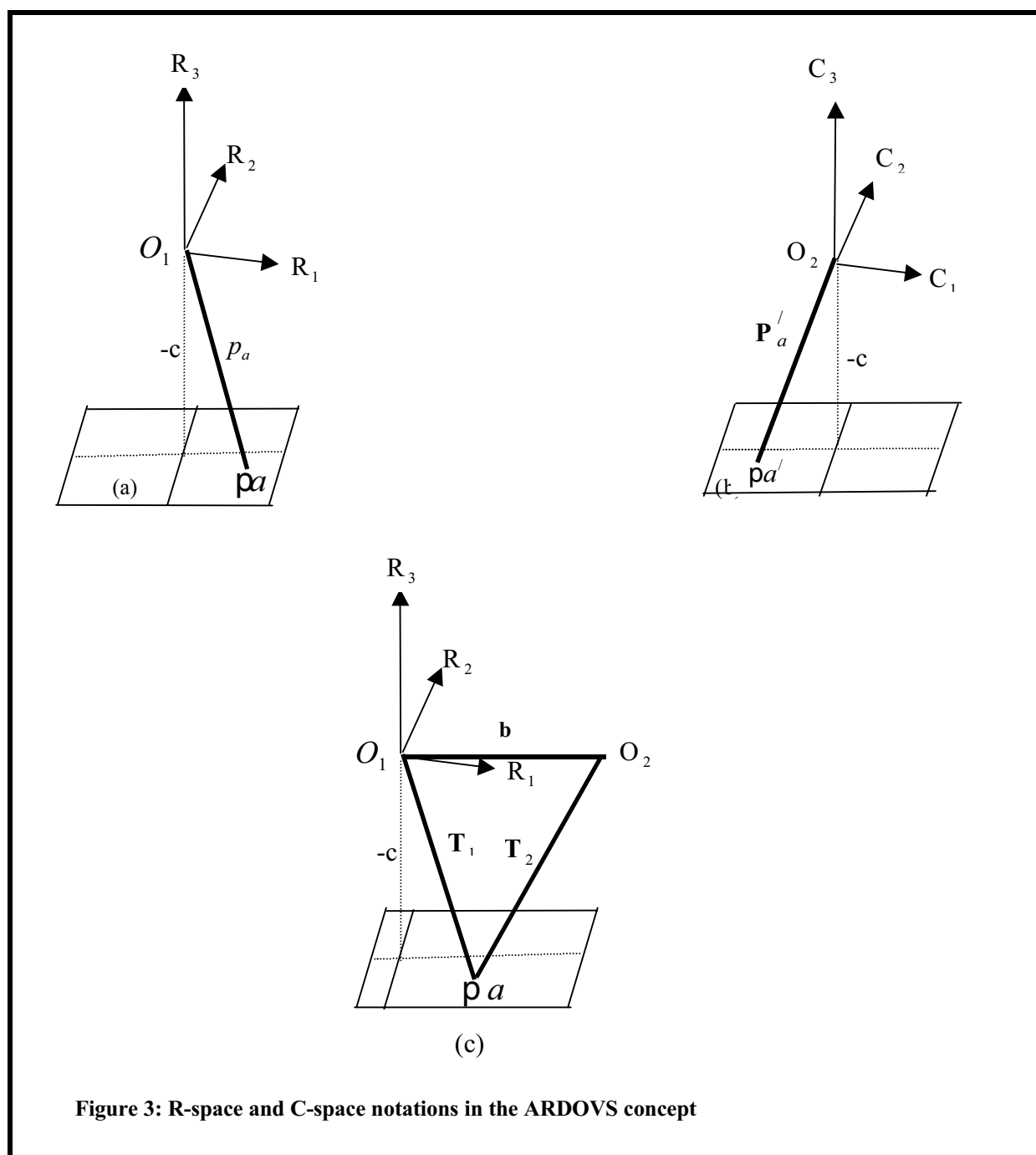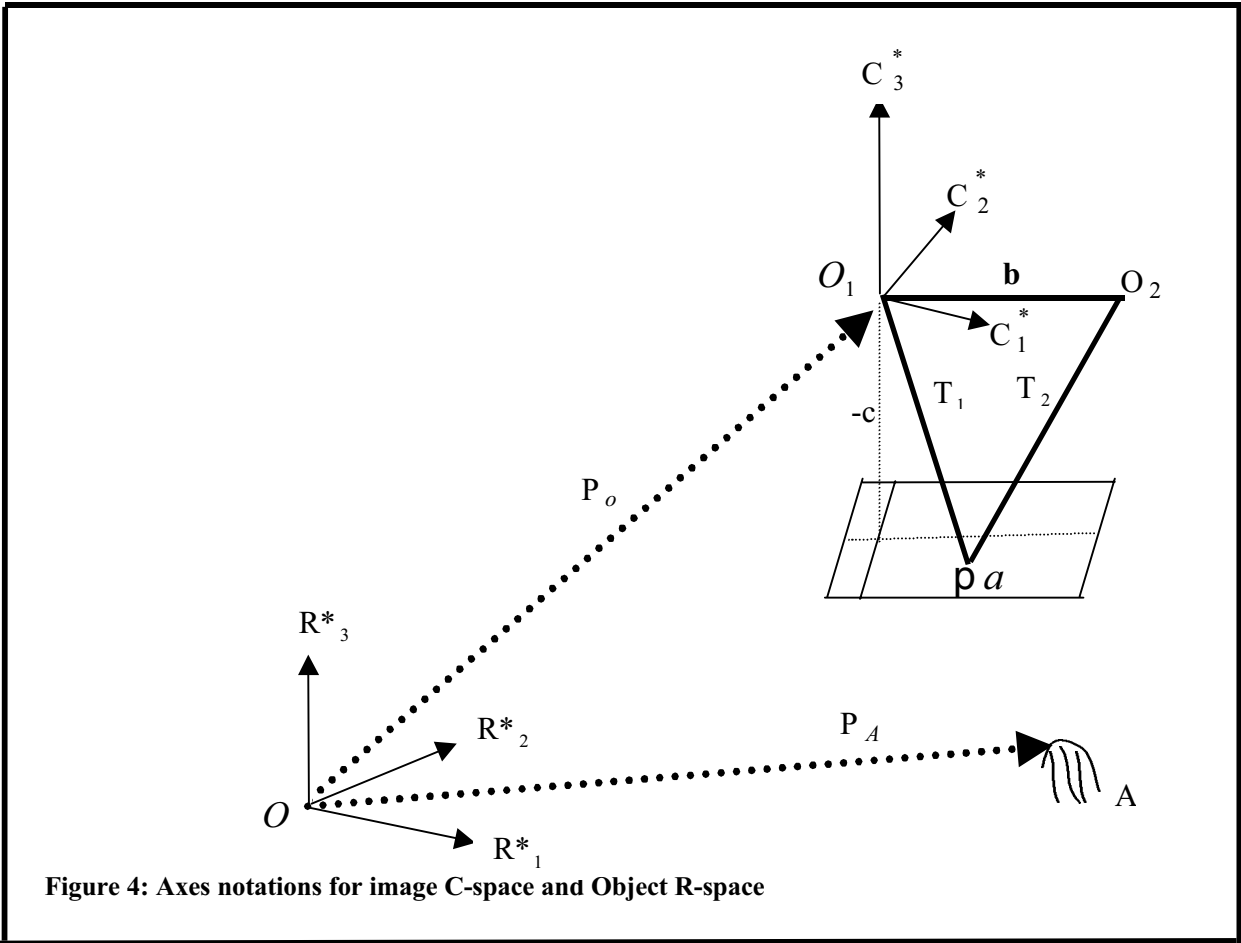

Figure 3: R-space and C-space notations in the ARDOVS concept

Figure 4: Axes notations for image C-space and Object R-space

2. ALGORITHM FOR REDUCTION TO THE OBJECT SPACE

The problem is to rigorously combine a pair of conjugate image space vectors to product the corresponding object space position vector. The operations involved in the forward reduction of a stereogram may be summarized as follows: vector transformation from an image **C**-space to an image **R**-space and by axes relabelling to an object **C**-space to an object **R**-space. This is achieved by application of the **ARDOVS** operational rules 1&2 to the tight image space vector (Pa') in Figure 5. This produces the left image space (**R**-space) vector polygon in figure 5), which is rigorously adjusted to minimize the parallax vector and to produce an equivalent image **R**-space element **T**. The axes of the image **R**-space are relabelled as an object **C**-space and a further application of rules 1 & 2 projects the vector element **T** onto the object **C**-space and a further application of rules 1 & 2 projects the vector element **T** onto the object space axes to produce the required object space information (see Figure 6). The full theoretical development of this algorithm is treated in Olaleye (1992).
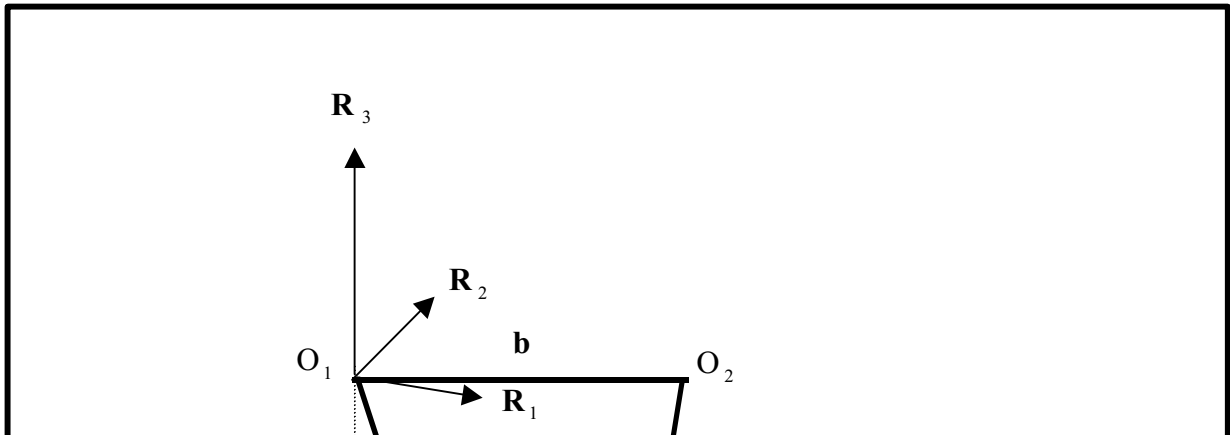
Figure 5: Image space intersection and parallax vector

Let

$$P_a = \begin{bmatrix} x_a \\ y_a \\ -c \end{bmatrix}, \qquad P'_a = \begin{bmatrix} x_a \\ y'_a \\ -c \end{bmatrix} \tag{2}$$

Using the **ARDOVS** operational rules 1 & 2, an element Pa' of the right image space (**C**-space)
translates to the image space (**R**-space) element as:

$$g = b + \begin{bmatrix} R_1.P'_a \\ R_2.P'_a \\ R_3.P'_a \end{bmatrix} \quad \text{or} \quad g = b + T_2 \tag{3}$$

where:

$$T_2 = \begin{bmatrix} R_1.P'_a \\ R_2.P'_a \\ R_3.P'_a \end{bmatrix} \tag{4}$$

b is the vector connecting two camera stations (i.e. photobase) and has the components

$$b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \quad \text{in the } \mathbf{R}\text{-space} \tag{5}$$

Naturally, in the **ARDOVS** theory, an element of the left image sees only the natural direction axes, therefore, we may represent the conjugate left image element as :

$$T_1 = \begin{bmatrix} i.P_a \\ j.P_a \\ k.P_a \end{bmatrix} \tag{6}$$

Theoretically, vector **g** (eqn.3) **should be equal to vector** T_1 in the **R**-space, i.e. the vector triangle in Figure 5 should close as shown in Figure 3c. This implies that the vectors $\mathbf{T}_1$ and T_2 should intersect at a point. However, in reality, they do not intersect at a point due to some residual parallax as shown in figure 5. Therefore, we employ mathematical optimization strategy to stretch the vectors $\mathbf{T}_1$ and T_2 along their direction to a point at which the parallax vector **P** is of minimum possible length. This is done in the **R**-space.

Let **P** be the parallax vector signifying the want of intersection of the two conjugate rays (dashed line in Fig. 4). Also, let the unit vectors corresponding to $\mathbf{T}_1$, and $\mathbf{T}_2$ be represented by $\overline{T}_1$ and $\overline{T}_2$ respectively, then from the vector polygon in Fig 4. we can write the vector equation

$$P = b - s_1\overline{T}_1 + s_2\overline{T}_2 \tag{7}$$

We employ the least squares optimization technique to determine values for s_1, s_2 which minimize the length of vector **p.** For this we formulate a vector space functional involving **P** whose stationary values yield the required values for s_1 and s_2. Using the inner product vector functional we have :

$$\frac{\partial(P.P)}{\partial s_1} = 0$$

$$\frac{\partial(P.P)}{\partial s_2} = 0$$

substituting for **p** from (7) provides

$$\frac{\partial(b - s_1\overline{T}_1 + s_2\overline{T}_2).(b - s_1\overline{T}_1 + s_2\overline{T}_2)}{\partial s_1} = 0$$

$$\frac{\partial(b - s_1\overline{T}_1 + s_2\overline{T}_2).(b - s_1\overline{T}_1 + s_2\overline{T}_2)}{\partial s_2} = 0 \tag{8}$$

Employing vector differential operators (Olaleye 1992), we obtain the normal equations from (8) as

$$\begin{bmatrix} \overline{T}_1.\overline{T}_1 & -\overline{T}_1.\overline{T}_2 \\ -\overline{T}_2.\overline{T}_1 & \overline{T}_2.\overline{T}_2 \end{bmatrix} \quad \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \overline{T}_1.b \\ -\overline{T}_1.b \end{bmatrix} \tag{9}$$

but $\overline{T}_1$ and $\overline{T}_2$ are direction vectors, so that (9) may be written as:

$$\begin{bmatrix} 1 & -\overline{T}_1.\overline{T}_2 \\ -\overline{T}_2.\overline{T}_1 & 1 \end{bmatrix} \quad \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \overline{T}_1.b \\ -\overline{T}_2.b \end{bmatrix} \tag{10}$$

The determinant of the normal equation (10) is given by

$$D = \begin{bmatrix} 1 & -\overline{T}_1.\overline{T}_2 \\ -\overline{T}_2.\overline{T}_1 & 1 \end{bmatrix} \equiv 1 - \left(\overline{T}_1.\overline{T}_2 \right)^2 \tag{11}$$

which is non- Zero unless the two rays are parallel. Then the solution to (10) is

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} 1 & \overline{T}_1.\overline{T}_2 \\ \overline{T}_2.\overline{T}_1 & 1 \end{bmatrix} \begin{bmatrix} \overline{T}_1.b \\ -\overline{T}_2.b \end{bmatrix} \tag{12}$$

The values of s_1, s_2 obtained from 12 provide the minimum length of the want of intersection of the conjugate rays. In practice, the two rays are assumed equally weighted so that we may locate the actual point of their intersection as being midway along vector **p**. Hence, the vector corresponding to this point of intersection may be written as:

$$T = \mathrm{T} = s_1 \overline{T}_1 + \frac{1}{2} P$$

or by substituting (7) for **p** and simplifying, we obtain

$$\mathrm{T} = \frac{b + s_1 \overline{T}_1 + s_2 \overline{T}_2}{2} \tag{13}$$

Applying the **ARDOVS** operational rules 1&2 (and noting that the axes of the image **R**- space are now relabelled for an object **C**-space), the object space location represented by the resulting vector **T** may be computed by the following vector product equation (See Olaleye 1992):

$$P_A = P_o + s \begin{bmatrix} R_1^*.T \\ R_2^*.T \\ R_3^*.T \end{bmatrix} \tag{14}$$

where

R_1^*, R_2^*, R_3^* are the apparent direction vectors of the object **R**-space. These are obtained by evaluating the vector expression in (6) using the computed absolute orientation elements.

s is the scale factor for the projection also determined by absolute orientation computation and

P$_o$ is the object space vector which locates the origin of the image space, also obtained from absolute orientation computation.

Substituting (7) in (8) we have

$$P_A = P_o + \frac{s}{2} \begin{bmatrix} R_1^*.(b + s_1\overline{T_1} + s_2\overline{T_2}) \\ R_2^*.(b + s_1\overline{T_1} + s_2\overline{T_2}) \\ R_3^*.(b + s_1\overline{T_1} + s_2\overline{T_2}) \end{bmatrix} \tag{15}$$

In order to further explore the geometric significance of this equation,

Let $\quad s^* = \frac{s}{2}, s_1^* = s^* * s_1, s_2^* = s^* * s_2 \tag{16}$

applying the distributive property of inner products of vectors to equation 9 and simplifying we have:

$$P_A = P_o + s^* \begin{bmatrix} R_1^*.b \\ R_2^*.b \\ R_3^*.b \end{bmatrix} + s_1^* \begin{bmatrix} R_1^*.\overline{T_1} \\ R_2^*.\overline{T_1} \\ R_3^*.\overline{T_1} \end{bmatrix} + s_2^* \begin{bmatrix} R_1^*.\overline{T}_2 \\ R_2^*.\overline{T}_2 \\ R_3^*.\overline{T}_2 \end{bmatrix} \tag{17}$$

This may be put in the form:

$$P_A = P_o + B + s_1^* \begin{bmatrix} R_1^*.\overline{T_1} \\ R_2^*.\overline{T_1} \\ R_3^*.\overline{T_1} \end{bmatrix} + s_2^* \begin{bmatrix} R_1^*.\overline{T}_2 \\ R_2^*.\overline{T}_2 \\ R_3^*.\overline{T}_2 \end{bmatrix} \tag{18}$$

Where the vector **B** is the equivalent of the airbase for the stereogram.
By relating equation 20 to Figures 3 & 4, it is evident that the forward reduction of conjugate image space vectors is achieved by a series of inner products of vectors. Equation 20 applies to every point appearing in the stereogram.

Computational Steps:

1. Construct the apparent direction vectors of the image **R-space and the object** R-space i.e.**R**$_1$, **R**$_2$, **R**$_3$ and R_1^*, R_2^*, R_3^* using the parameters of R.O. & A.O. respectively. Also compute the following (note that P$_a$ and **P**$_a$' are conjugate image space vectors)

$$P_a = \begin{bmatrix} x_a \\ y_a \\ -c \end{bmatrix}, \; P^{/a} = \begin{bmatrix} x_a \\ y'_a \\ -c \end{bmatrix}, \; b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \tag{19}$$

$$T_1 = \begin{bmatrix} i.P_a \\ j.P_a \\ k.P_a \end{bmatrix}, \; T_2 = \begin{bmatrix} R_1.P^{/}_a \\ R_2.P^{/}_a \\ R_3.P^{/}_a \end{bmatrix}$$

$$\overline{T}_1 = \frac{T_1}{|T_1|}$$

$$\overline{T}_2 = \frac{T_2}{|T_2|}$$

$$D = 1 - \left(\overline{T}_1.\overline{T}_2 \right)^2$$

$$s_1 = \frac{\overline{T}_1.b - (\overline{T}_1.\overline{T}_2)\overline{T}_2.b}{D}$$

$$s_2 = \frac{(\overline{T}_1.\overline{T}_2)\overline{T}_1.b - \overline{T}_2.b}{D}$$

$$s^* = \frac{s}{2}, s_1^* = s^* * s_1, s_2^* = s^* * s_2$$

evaluate R_1^*, R_2^*, R_3^* from equation 6 and the absolute orientation parameters

$$B = s^* \begin{bmatrix} R_1^*.b \\ R_2^*.b \\ R_3^*.b \end{bmatrix} \tag{20}$$

2. The object space coordinate vector is then computed as

$$P_A = P_o + B \ + \quad s_1^* \begin{bmatrix} R_1^* . \overline{T_1} \\ R_2^* . \overline{T_1} \\ R_3^* . \overline{T}_1 \end{bmatrix} \ + \ s_2^* \begin{bmatrix} R_1^* . \overline{T}_2 \\ R_2^* . \overline{T}_2 \\ R_3^* . \overline{T}_2 \end{bmatrix} \tag{21}$$

3.   Repeat steps 1-2 for all points

3.   Algorithm for object space to conjugate image spaces

Conceptually, the task here is to resolve the object space vector element into its left and right image space vector elements. The operations involved may be summarized as follows: vector transformation from an object **R**-space to an object **C**-space (and through axes relabelling) to an image **R**-space to an image **C**-space. This is achieved by application of the **ARDOVS** operational rules 1 & 3 to the object space element (P_A) in Figure 5. This produces an object **C**-space element **T** (Figure 5). The axes of the object **C**-space are relabelled as an image **R**- space and a further application of rules 1 & 3 projects the vector element **T** onto the left and right image spaces to produce the conjugate image space vectors (See figure 6). Note that the appropriate scales are applied to the projection. The theoretical development of this algorithm is treated in Olaleye (1992). The algorithm is listed below:
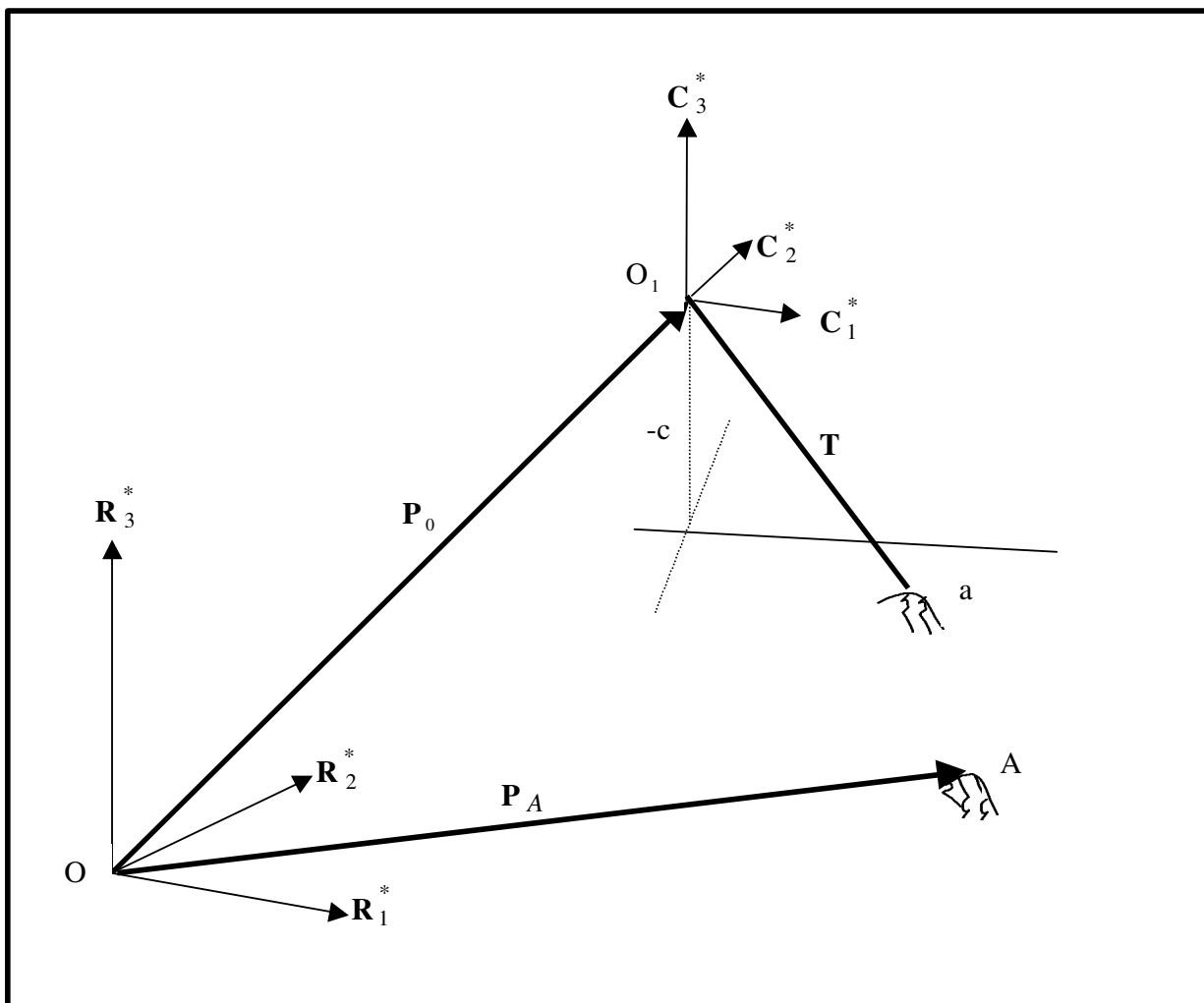
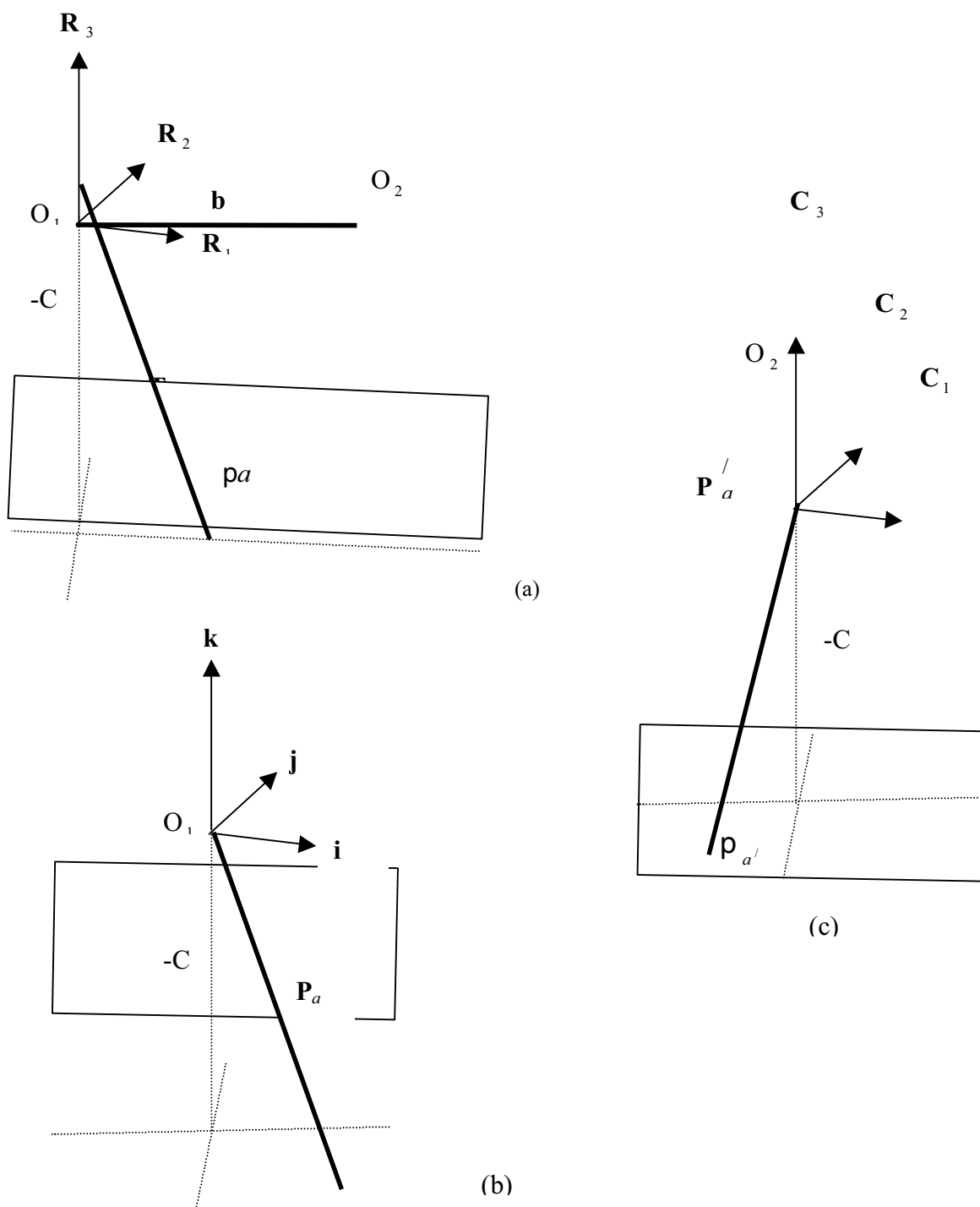**Figure 5: Object R-space to object C-space Transformation**

(a)

(b)

(c)

Figure 6: Image R-space to conjugate image C-shape Transformation

Applying the **ARDOVS** operational rules 1 & 2 we may write:

$$T = \frac{1}{s}\begin{bmatrix} C_1^*.(P_A - P_0) \\ C_2^*.(P_A - P_o) \\ C_3^*.(P_A - P_o) \end{bmatrix}$$

(22)

$$s_1 = \frac{c}{k.T}$$

$$s_2 = \frac{c}{C_3.(T-b)}$$

Where s is the scale from the object to image space obtained from absolute orientation computation. s1 and s2 are the image space scales.

For the left image we have:

$$P_a = s_1 \begin{bmatrix} i.T \\ j.T \\ -k.T \end{bmatrix} \tag{23}$$

and for the right image we have

$$P_a^/ = -s_2 \begin{bmatrix} C_1.(T-b) \\ C_2.(T-b) \\ C_3.(T-b) \end{bmatrix} \tag{24}$$

where C_1, C_2, C_3 are obtained by evaluating equations 2 with the relative orientation information C_1^*, C_2^*, C_3^* are obtained by evaluating equations 2 with he absolute orientation information.

Computational steps

1. Compute C_1, C_2, C_3 and C_1^*, C_2^*, C_3^* from R.O and A.O elements.
2. Compute

$$T = \frac{1}{s} \begin{bmatrix} C_1^*.(P_A - P_0) \\ C_2^*.(P_A - P_o) \\ C_3^*.(P_A - P_o) \end{bmatrix} \tag{25}$$

$$s_1 = \frac{c}{k.T}$$

$$s_2 = \frac{c}{C_3.(T-b)}$$

Let image vector:

$$P_a = s_1 \begin{bmatrix} i.T \\ j.T \\ -k.T \end{bmatrix}$$

and for the right image we have

$$P_a^{/} = -s_2 \begin{bmatrix} C_1.(T-b) \\ C_2.(T-b) \\ C_3.(T-b) \end{bmatrix} \tag{26}$$

4. CONCLUSION

It is obvious from the presentation that the vector space approach offers many advantages. Formulations which otherwise look cumbersome are made very compact and incitefully appealing. Computational examples will be given in the second part of this paper.

5. REFERENCES:

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