DEVELOPING A THREE-DIMENSIONAL TOPOLOGICAL DATA MODEL

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ABSTRACT

In this study, a topological data model is defined based on cell-complexes to represent the geometry of spatial objects and their topological relations. It is discussed how a spatial object can be fragmented into simple cells and represented by a regular cell-complex. Such a cell-complex can describe both the geometry and internal topological structure of a spatial object. In other words, the regular cell-complex is both an approximation of the geometry of the spatial object and a topological equivalent of it. Topological relations between n-dimensional ($0 \leq n \leq 3$) spatial objects will be defined based on the integration (overlay) of regular cell-complexes representing spatial objects in a 3-dimensional space R^3. The result is a more general cell-complex called Generalized Singular (GS) cell-complex, which consists of cells of different dimensions.

The direct consequence of integrating spatial objects is the intersection of their cells. Some types of singularity, which are caused by the intersection of cells, along with the way they should be handled are discussed. The only type of singularity we allow in the GS cell-complex happens when a higher-dimensional cell contains lower-dimensional cells that are not part of its boundary cycles. The reason why we should not allow other types of singularity is discussed. Finally, the rules governing the GS cell-complex are described.

1 INTRODUCTION

Compared to the traditional maps, spatial analysis of thematic data is improved efficiently by digital co-registration and overlaying analysis capabilities of 2D GISs. Many analyses, predictions and simulations that never could be done using maps is now possible using 2D GISs. However, many applications require the modeling and analysis of 3D phenomena, which are not properly supported by 2D GISs. Many systems have already been developed for modeling spatial objects and simulating their movements and behavior in 3D space. However, because of their shortcoming in representing the topological relations between objects, such systems are not capable of many spatial analyses that are expected from a GIS. Therefore, there is a need for a 3D topological data model to represent both the 3D geometry and topological relations of spatial objects.

Developing such a data model is not a trivial and simple extension of the available 2D topological data models. The geometry of spatial objects can be much more complicated in a 3D than in a 2D space. In addition, the relations between spatial objects in 3D space are of more types. Development of a 3D spatial data model should be based on the study of geometry and relations of spatial objects in 3D space.

2 CELLS, BUILDING BLOCKS OF SPATIAL OBJECTS

The first applications of cells and cell-complexes were in topology, which is the most general form of geometry. Topologists use cells and cell-complexes for the fragmentation of spaces they study. This fragmentation is to simplify the calculation of very general characteristics (topological properties) of those spaces. With the development of computer science and technology, several systems such as CAD/CAM, Computer Graphics and GIS emerged to represent spatial objects for different applications. Such spatial objects are usually complex. To overcome this complexity, in those systems also, spatial objects are fragmented and represented as cell-complexes composed of simple primitives (cells) of different dimensions. In this research we follow the same approach and fragment spatial objects into primitive cells of proper dimensions.

In the followings, we use some terms from topology such as manifold, manifold with boundary, *n*-disk and topologically equivalent without defining them. For the definitions of these terms, reader can refer to Croom (1978), Singer and Thorpe (1967) and Stillwell (1980). We define a *Generalized Regular n-dimensional cell* (*GR n-cell*) as a connected part of a Euclidean *n*-manifold (*n*-dimensional manifold) with one or more subdivided (*n*-1)-manifold boundary cycles ($2 \le n \le 3$). We exclude the 0-cell from this definition simply because we can not assume a 0-manifold with subdivided (-1)-dimensional manifold boundary cycles. A GR 0-cell is simply defined as a single point. A 1-cell is also excluded from that definition for the same reason; i.e. we do not have any 0-manifold to be the boundary of the 1-cell. A 1-cell can be defined as a connected part of a Euclidean 1-manifold with exactly two 0-cells as its boundary. It means that 1-cells and 2-cells do not have any curvature or folding. A 1-cell is a straight segment and a 2-cell is a part of a flat surface possibly with some inner boundaries. In the rest of the thesis, whenever it is more convenient, we use the more familiar terms of *node, arc, facet* and *body* instead of GR 0-cell, GR 1-cell, GR 2-cell and GR 3-cell respectively.

The definition of GR cell includes any connected part of an Euclidean *n*-manifold with the only condition that any of its boundary cycles should be a subdivided (*n*-1)-manifold. We use the term *subdivided* to note that such a manifold boundary is also represented by a generalized regular (*n*-1)-dimensional cell-complex (we define these terms latter), which actually subdivides the boundary (*n*-1)-manifold into a bunch of cells. We name our cell *generalized* because it extends and incorporates the simplicial and regular CW complexes used in topology. In addition, we call it *regular* because the subdivided manifold boundary cycles form the (*n*-1)-skeleton of the cell and so are regular. Figure 1 shows some examples of cells that are in accordance with the definition of GR cell.

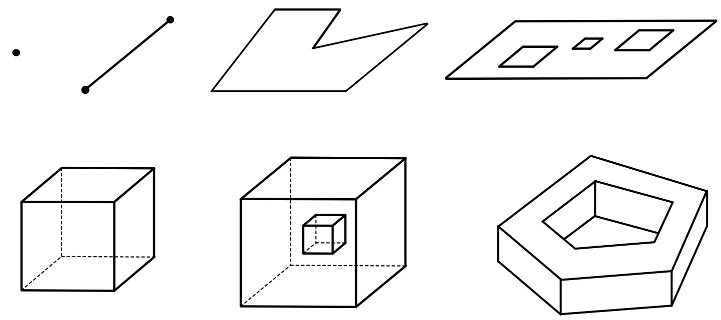


Figure 1: Some examples of the Generalized Regular cells

3 SPATIAL OBJECTS REPRESENTED AS REGULAR CELL-COMPLEXES

3.1 Spatial Objects

Usually, we use the term *spatial objects* for any object or phenomenon that spatially extends through the 3-dimensional space (the physical world we live in). Basically, such an object, no matter how thin and small, occupies some parts of the 3-dimensional space ant so inherent the dimension from that space. It means that, all spatial objects we are dealing with are actually 3-dimensional. However, sometimes a spatial object is much more extended in one or more dimensions than in the others. For example, although a water supply pipe is a cylinder (a 3-dimensional object) and has volume, but still its diameter is negligible in comparison with its length. Therefore, in spatial data modeling the image of such object can be considered as a 1-dimensional spatial object. Similarly, a spatial object can be considered and represented as a 2-dimensional object. An object that is small enough, regarding the scale or in comparison with other objects around it, can be considered as a 0-dimensional spatial object. Therefore, in the followings, whenever we say that a spatial object is *n-dimensional*, $0 \le n \le 2$, we mean that the extension of different parts of it along one or more dimensions are insignificant.

To be more clear and specific about the concepts that will be discussed in the following, we assume a spatial object to be *uni-dimensional*, i.e. composed of parts of the same dimension. In reality, an object can be *multi-dimensional*, i.e. composed of parts of different dimensions. If an object is multi-dimensional, it can be divided into some uni-dimensional spatial objects. For example, a house with all its assets is a multi-dimensional object that can be divided into some 3D, 2D and 1D spatial objects. Later, all these uni-dimensional spatial objects can be aggregated to build up the multi-dimensional object of the house. Figure 2 represents some examples of uni-dimensional spatial objects.

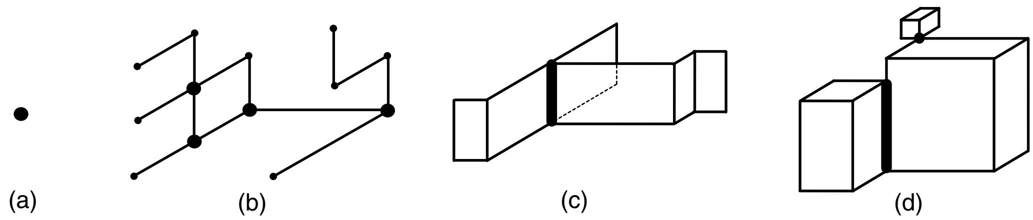


Figure 2: Examples of uni-dimensional spatial objects: a 0-dimensional spatial object is represented by a single node. Spatial objects of dimensions 1, 2 and 3 are represented as GR cell-complexes, in which the points with non-manifold neighborhood are highlighted. Because of such non-manifold points, none of these objects can be represented by a single cell.

3.2 Generalized Regular Cell-Complex

Any such spatial object, although uni-dimensional, still can be geometrically complex and might have complex topological relations with other objects. To overcome those complexities, any such uni-dimensional spatial object is fragmented into GR cells and represented by a cell-complex consisting of those GR cells. Usually a cell-complex is defined in the same way as a simplicial complex. We define a G*eneralized Regular* (GR) cell-complex *U* as a set of GR cells if the following conditions are satisfied:

- The set *U* is finite.
- If *U* contains a GR cell then it also contains all the faces of this cell.
- The intersection of two GR cells of *U* is either empty or a common face of them.

3.3 Geometry of Spatial Objects and handling Curvature and Folding

Although topology is also a part of geometry, we exclusively define the geometry of cells and objects as their shape, size, orientation and positional information. The geometry of a spatial object can be determined based on the geometry of cells in the cell-complex representing it. The geometry of each cell is also determined from the geometry of its lower-dimensional boundary cells, down to the lowest-dimensional cells (nodes). Therefore, all the shape, size, orientation and positional information about objects are stored in the coordinates of their nodes.

In the definition of GR cell given by Pigot (1995), he allowed a cell to be a topologically equivalent of a part of the Euclidean manifold. With such a definition, an arc or facet can be curved and/or folded and still be topologically equivalent to a part of the Euclidean 1-manifolds or 2-manifolds. It is obvious that curvature and folding are meaningless for nodes and bodies in a three-dimensional space. Allowing the curvature and folding in an arc or facet, its geometry can not be always defined simply based on the geometry of its boundary cells. Only in the case of a folded arc, it seams possible to use the same method as used in 2-dimensional GISs. That is, to have a list of coordinates associated with the arc to represent the intermediate points between the two end nodes of the arc.

Our intention in using cells and cell-complexes is to represent not only the topological relations between spatial objects, but also their geometrical structure. Therefore, we modified our definition of GR cell by allowing it to be only a part of the Euclidean manifold and not a topological equivalent of it. Such a definition explicitly limits the arcs and facets to be straight and flat respectively.

3.4 Subdividing a Spatial Objects into GR Cells

With the above results, we can determine how and where a spatial object should be fragmented in order to be represented by a set of simple cells. As is shown in Figure 2(a), a 0-dimensional spatial object can be simply represented by a single node. A 1-dimensional spatial object should be fragmented at any point with a neighborhood that is not topologically equivalent to a 1-disk. Such points are where three or more line-segments are intersecting. In the Figure 2(b), such *non-1-manifold* points are highlighted. Also, for the sake of consistency and homogeneity, we selected not to use the list of intermediate point coordinates, explained above. Therefore, at any folding point the object will be fragmented. The curvature can be approximated by a set of arcs. A 2-dimensional spatial object is also shown in Figure 2(c). Similar to 1-dimensional case, such an object is fragmented at the non-2-manifold points, which are depicted in the Figure as a highlighted arc. In addition, it is fragmented at any folding line. A Curved facet can be replaced by a collection of flat facets. Finally, a 3-dimensional object can be subdivided at any non-3-manifold point such as those highlighted in the Figure2(d).

4 TOPOLOGICAL RELATIONS BETWEEN CELLS AND BETWEEN CELL-COMPLEXES

Topological relations can be divided into two types, the relations between the cells in a cell complex and the relations between different cell complexes representing spatial objects. What we need for spatial analyses and queries in a GIS are the topological relations between spatial objects (cell complexes) themselves. Studies done by Egenhofer and Herring (1991), Egenhofer and Mark (1995) and others show how numerous and complex the topological relations between spatial objects are.

Usually, to calculate the relations between spatial objects, in 2D-GIS the union of cell complexes representing spatial objects is calculated. It means that, the cell complexes representing spatial objects are embedded together in a Euclidean 2-manifold and the topological relations between their cells are calculated again. Then the relations between spatial objects can be derived from the relations among their cells. For example, when we want to query if a river is going through a forest, we only need to check if any of the arcs representing the river has the facets of the forest as both its co-boundaries. We select the same approach as the above. GR cell-complexes representing spatial objects of different dimensions are embedded together (overlaid) in a 3-dimensional Euclidean manifold. The Euclidean 3-dimensional manifold represents the 3-dimensional space of spatial objects. The result is a more general cell complex, which does not obey the rules described in the definition of the GR cell-complexes. We call it a G*eneralized Singular (GS) cell-complex* in contrast to GR cell-complexes. The definition of such a cell-complex and its difference with GR cell-complex will be discussed later. The immediate consequence of such integration is that cells of different spatial objects might intersect each other. So, whenever the interior of a cell is crossed (cut) by another cell, at the intersection points new cells will be created (if such a cell does not exist already) and one or both of intersecting cells will be subdivided. Then simple topological relations between all cells in this GS cell-complex should be calculated. The result is that, latter, complicated topological relations between spatial objects can be derived from simpler topological relations among their cells in the GS cell-complex.

5 OVERLAY OF SPATIAL OBJECTS

As was explained, when overlaying GR cell-complexes representing spatial objects some cells intersect each other and sometimes in the intersection points *new cells* should be created. For example when two arcs are intersecting in their interiors, in the intersection point a new node is created; also, when an arc is intersecting the interior of a facet, in the intersection point a new node is created. Usually the creation of such new cells have two consequences. Any of the two intersecting cells either will contain the new cell or will be subdivided by the new cell.

The criterion for any intersecting cell to be subdivided or not is its *continuity* after the intersection. Here, we call a k-cell *continuous* if we can draw a path between any two points of it, completely inside it and without crossing any j-cell inside it ($j < k$). If by the creation of a new cell at the intersection points, the intersecting cell remains continuous then it will not be subdivided and the new cell will be embedded in it. For example, when an arc is crossing through the interior of a facet, i.e. when they intersect in one point in their interiors, at the intersection point a new node will be created. Having such a new node created, the arc will not be continuous anymore and will be subdivided into two smaller arcs adjacent in the new node. In contrast, the facet will remain continuous. Therefore, it will not be subdivided by the node and the node will be contained in it. The relation between the two new arcs (subdivision of the first arc) and the facet is based on the new intersection node, which is both a common boundary of the arcs and a dangling node inside the facet. When two cells intersect, it is not always necessary to create new cells in the intersection points. Sometimes the intersection is in an existing cell, which might be part of the boundary of one or both of the intersecting cells. For example, a facet can touch the interior of another facet, i.e. a boundary arc/node of one facet can be in the interior of the other one. Calculating the union (overlay) of spatial objects represented as GR cell-complexes should be based on the following general rules:

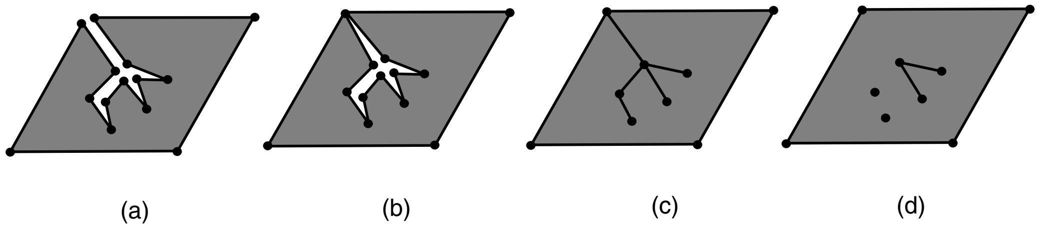
- The intersection of two cells should always be a cell (or some cells) of the cell-complex.
- If the intersection of two cells is not of the same dimension as both of them and already a cell exists at the intersection of the two cells, then we only need to define the relation of the intersection cell with the two intersecting cells. An example is when two cells intersect in a part of the boundary of one or both of them)
- If the intersection of two cells is not of the same dimension as both of them and there is not any cell at the intersection of the two cells (i.e. if the two cells intersect each other in their interiors), then at the intersection points new cell (cells) should be created. By the creation of the new cell, if any of the two intersecting cells become discontinuous, then that cell should be subdivided into two smaller cells, which are adjacent to each other at the new cell. Otherwise, the new cell will be contained inside that intersecting cell.
- Two cells that are equal, i.e. cover each other completely, will be represented as only one cell in the GS cell-complex.
- One cell can not be completely covered by (contained in) another cell of the same dimension. In such a case, the containing cell will be subdivided into two parts that are shared and not shared with the smaller cell. The two shared (matching) parts will be equalized (represented by one cell). Such a subdivision might cause the creation of cyclic singularity, which will be explained latter. However, a cell can be completely contained in another cell of a higher dimension without demolishing the continuity of the higher-dimensional cell, e.g. an arc can be inside a facet but a facet can not be inside another facet.
- If the intersection of two cells is not equal to any of them but is of the same dimension as both of them, e.g. when two bodies overlap each other in a part of their volumes, then the two cells should be subdivided such that the intersection part becomes a new and separated cell.

6 SINGULARITIES RESULTED FROM THE OVERLAY OF SPATIAL OBJECTS

When we have the union of spatial objects of the same dimension, the result of intersections is another cell-complex composed of cells of the same dimension, with the only difference that they are more subdivided. For example, when cell-complexes composed of 2-cells and representing 2-dimensional spatial objects are combined, the result of intersection is another cell-complex in which the 2-cells are subdivided into smaller 2-cells. Sometimes, we have the integration of spatial objects of different dimensions, which are represented as cell-complexes with cells of different dimensions. The result of integration will be a cell-complex consisting of multi-dimensional cells. So, for example, there might be 2-cells that have dangling and/or internal 1-dimensional and 0-dimensional cell-complexes. Such situations are referred to as *singularity*.

Obviously, those dangling and isolated internal cell-complexes are not part of the boundary of the 2-cell. They simply are parts of cell-complexes representing other spatial objects. Despite this fact, in early works of Corbett (1975) and Corbett (1979), such dangling and internal cells were considered as part of the boundary of the cell in order to describe the neighborhoods of cells in such singularities. Corbett (1979) defined a *regular cell* as the image of a one-to-one, onto and continuous transformation (homeomorphism) applied to both the interior and boundary of the *n*-disk (*n*-ball). In contrast to this, he defined a *singular cell* as an image of the *n*-disk, in which the one-to-one condition of the transformation is relaxed. The consequence of this relaxation is that some points in the boundary of the n-cell can be identified. In other words, the boundary of a singular cell is the image of a continuous mapping applied to the boundary cycle of a regular cell that results in the identification of sets of points. The effect of such identifications is *singularity*, i.e. singularity is the set of points whose neighborhoods are no longer homeomorphic to the 1-disk. He distinguishes among three types of singularities for 2-cells: *cyclic, acyclic* and *interior*.

Figure 3 shows some examples of such singularities. In Figure 3(a), we have a regular 2-cell, which has a boundary cycle homeomorphic to the boundary of a 2-disk. Notice that the boundary of a 2-disk is a 1-dimensional sphere, which is a 1-manifold. Figure 3(b) shows a mapping applied to the regular cell that identifies two points in the boundary of the regular cell. The result is a 2-cell with a cyclic singularity, i.e. it has a new 1-cycle as part of its boundary and attached to its outer boundary. The problem with such a boundary is that in the identified points the boundary is not a 1-manifold anymore. Similarly, by applying another continuous mapping and identifying some other points in the boundary of the 2-cell Figure 3(c) will be resulted. This Figure shows a 2-cell with acyclic singularity, which is a 2-cell with an internal tree structure composed of 1-cells. Finally, by the identification of more points Figure 3(d) is resulted, which shows a 2-cell with interior singularity.



Figiure 3: Examples of singularity: (a) a regular 2-cell, (b) a 2-cell with cyclic singularity, (c) a 2-cell with acyclic singularity and (d) a 2-cell with interior singularity.

Corbett (1979) assumed singularities as part of the boundary of the singular cell by allowing the boundary of a singular cell to be a *pseudomanifold*. A pseudomanifold is a figure in which almost all points obey the manifold conditions and is formed by gluing (identifying) some points of a manifold. Pigot (1995) followed the same approach and defined a *generalized singular k-cell* as a Euclidean k -manifold with i subdivided (k-1) -pseudomanifold boundary cycles ($i \geq 1$). The j-cells ($j \leq k$-1) in the subdivided (k-1)-pseudomanifold boundary cycles are a subset of the (k-1)-skeleton. Having defined such a singular cell, he defined the generalized singular cell-complex as a complex composed of such singular cells. Such a cell-complex is the union of cell-complexes representing spatial objects.

In this research, we take a different approach. In our model, we have only *GR cells* and not any *singular cell*. Instead, when integrating cell-complexes that represent spatial objects of different dimensions, the regular cells are allowed to have dangling or isolated cells of lower-dimensions in their interiors. So, in our approach dangling and internal structures are considered as being contained inside the higher-dimensional cell and not part of its boundary. Such an approach has two advantages compared with the other one: First, it is more intuitive and more representative of the real relation between the higher-dimensional cell and the dangling/internal cells inside it. We should remember that, essentially, the boundary is supposed to be the border between the interior and the exterior of an object or cell. However, when a k-cell has some dangling or internal j-cells ($j \leq k$-1) inside it, the whole k-dimensional neighborhoods of those j-cells are at the interior of the k-cell. Second, in our approach the boundary of a regular k-cell is enforced to be a (k-1)-manifold. As it will be explained latter, this manifold condition can be checked using the co-boundary orderings. In fact, this condition is the basis of topological consistency checks to differentiate between correct and incorrect cells (Mesgari et all, 1998). In the other approach, checking the condition of the boundary being a pseudomanifold is not as easy as in our approach. Because, basically, the class of pseudomanifolds is much more general than the class of manifolds and the neighborhood of points in singularities are more complex than regular points.

We explained that in our approach, the dangling and internal structures inside a cell are not assumed as part of its boundary and still the boundary of such a cell should be a manifold. However, in order to be consistent with other approaches to the problem of integrating spatial objects, still we refer to such situations as acyclic and interior *singularities*. Nevertheless, these situations are in contradiction to a general rule of regular cell-complexes (the rule that cells intersect only along their boundaries). Therefore, we refer to the multi-dimensional cell-complex resulted from the integration of spatial objects of different dimensions, which has such singularities, as the Generalized Singular cell-complex.

The two types of acyclic and interior singularities are allowed in our model by accepting a cell to be inside another cell of a higher dimension. In this regard, cyclic singularity is different from the other two types. The problem with this singularity is that in the identified points the boundary is not a manifold anymore, and so, the cell can not be a GR cell anymore. This type of singularity is usually created by the subdivision of a regular cell that is shared (covered) partly by a smaller cell of the same dimension, as explained in section 5. We eliminate cyclic singularities by subdividing such a singular cell into two new GR cells.

7 GENERALIZED SINGULAR CELL-COMPLEX REPRESENTING THE OVERLAY OF SPATIAL OBJECTS

From the above discussions, it can be concluded that the union of spatial objects can not be realized without accepting some types of singularities. We discussed how the overlapping and containment relations between cells (and consequently spatial objects) of the same dimension can be represented by subdividing them into shared and non-shared parts. The same can not be done for cells (and so objects) of different dimensions and as a result we encounter some types of singularities. From the rules described in section 5 and above discussions, the definition of the GS cell-complex, which is the result of integrating spatial objects of different dimensions, can be concluded. We define a *Generalized Singular (GS) Cell-Complex* as a set of generalized regular cells X if the following conditions are satisfied:

1. The set X is finite.
2. If X contains a cell, then it also contains all the faces of the cell.
3. Not only the boundary, but also the interior of a cell can be intersected by another cell, with the only condition that the intersection should be in some cells of X.
4. Any k-cell of X, although having singularities, is continuous, i.e. it should always be possible to draw a path between any two points of the k-cell, completely inside it, without crossing any j-cell inside it ($j < k$).
5. Two cells of X of the same dimensions can not intersect each other in a part of their interiors that is of the same dimension as the two cells.

We adopt the first two conditions from the regular cell-complexes. Condition 1 is simply to prevent the assumption of infinite number of cells; usually the cell-complexes we want to create in GIS are satisfying this condition. Condition 2 prevents the creation and existence of a cell without all its faces (boundaries and boundaries of boundaries etc.) existing. Before we create a cell, all its faces should be created and they should be part of the cell-complex. This condition also guarantees that we do not delete a cell when it is still part of the boundary of another cell. First, the higher-dimensional cell should be deleted and only then, a part of its boundary can be deleted. Condition 3 is to allow the acyclic and interior singularities, when the interior of the higher-dimensional cell is intersected by the other cell. Previously it was explained that the definition of the generalized regular cell does not allow the presence of cyclic singularities because the cells in GS cell-complex should be GR cells. Based on this condition, whenever arcs and facets cross each other, in the intersection points new cells should be created (if not existent already) and added to the cell-complex. Condition 4 is to enforce the continuity of cells. Previous condition enforces the existence or creation of a cell in the intersection of two cells. Condition 4 suggests that whenever any of the two intersecting cells becomes discontinuous by creating the new cell, we should subdivide that intersecting cell. Finally, condition 5 is to prevent the overlapping and containment relations between cells of the same dimension.

These five conditions along with the definition of GR cell are enough to check and ensure the correctness of the GS cell-complex such that it represents the topological relations between cells of spatial objects. In some other parts of this research, different types of topological relations between cells of the singular cell-complex are studied. Those topological relations are represented by a modified version of the cell-tuple data structure of Brisson (1990). The modifications are necessary to cover the singular situations. Also some topological operators are designed and are going to be formally specified for the construction of cells in the GS cell-complex. More explanation about these parts of the research is out of the scope of this article.

8 CONCLUSIONS

The geometry and topological structure of spatial objects of different dimensions were represented by regular cell-complexes of proper dimensions. To determine the topological relations between them, spatial objects are integrated in a more general cell-complex, which includes some types of singularities. Singularities are the result of intersection between cells of different dimensions in their interiors. Different types of singularities were studied. The rules governing the general cell-complex that represent the integration of spatial objects were determined.

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