Analysis of Topological Relations between Fuzzy Regions in a General Fuzzy Topological Space

Xinming Tang[1], Wolfgang Kainz[2]

International Institute for Geo-Information Science and Earth Observation (ITC), P.O. Box 6, 7500 AA Enschede, The Netherlands, Tang@itc.nl[1]
Cartography and Geoinformation, Department of Geography and Regional Research, University of Vienna, Universitätsstrasse 7,A-1010 Vienna, Austria, w.kainz@univie.ac.at[2]

Abstract

Topological relations are one of the most fundamental properties between spatial objects. The topological relations between crisp spatial objects have been well identified. However how to formalise the topological relations between fuzzy regions needs more investigation. The paper starts from introduction of boundaries defined in the fuzzy topological space. By use of a definition formally equivalent with the boundary of a crisp set in the crisp topological space, several novel notations of the fuzzy topological space are proposed. These notations are then proved to be topological properties. In order to investigate the topological relations, a 9-intersection matrix and a 4*4 intersection matrix are formalized based on different topological parts of two fuzzy sets. For the identification of the topological relations between two fuzzy spatial objects, a simple fuzzy region is defined topologically. By use of the 9-intersection matrix, 44 relations are identified. These relations can be further decomposed by use of the 4*4 intersection matrix. Since the analysis is based on the general fuzzy topological space, the results will be more applicable for GIS modelling.

1 Introduction

A topological relation is a relation that is invariant under homeomorphisms. Topological relations have an important significance in GIS modelling since they are the basis for spatial modelling, spatial query, analysis and reasoning. How to identify the topological relations between spatial objects is a critical point in GIS modelling.

During recent years, topological relations have been much investigated in the crisp topological space. White (1980) introduced the algebraic topological models for spatial objects. Allen (1983) identified thirteen topological relations between two temporal intervals. The famous 4-intersection approach (Egenhofer 1989, Egenhofer and Franzosa 1991) and the 9-intersection approach (Egenhofer and Herring 1990) were proposed for formalism of topological relations between two simple regions, which are defined in the crisp topological space. These approaches were then extended in different ways and different applications (Egenhofer et al 1994a, Egenhofer and Franzosa 1994, Herring 1991, Chen et al 2000, 2001). Most of these theoretical findings were applied in the commercial GIS software such as maintaining of topological consistency (Egenhofer and Sharma 1993, Egenhofer et al 1994b, Kainz 1995, Van Oosterom 1997) and spatial reasoning (Dutta 1991, Al-Taha 1992, Egenhofer 1994).

There are several other ways that can also be used for the identification of topological relations. Kainz et al (1993) investigated some of the topological relations based on poset and lattice theory. They pointed out that the poset and lattice can be also used for the GIS modelling which can reduce the complexity of some queries. The topological relations were also deduced based on logic (Randell et al 1992). Eight relations were identified based on their RCC theory. It is also feasible for their theory to be adopted for spatial reasoning (Cohn et al 1997, Bennett et al 1997, Bennett and Cohn 1999).

More recently the topological relations have been extended into fuzzy domains since the spatial features are not always crisp. Fisher (1996) provided a good example of fuzzy object by land cover classification from satellite images. Fuzzy objects should be accommodated into the conventional GIS models. Schneider (1999) worked on definitions of fuzzy spatial concepts such as fuzzy point, fuzzy line and fuzzy region. Cheng et al. (1997, 2001) modelled the processes of forming fuzzy objects by taking coastal zones as example. Molenaar (1996, 1998) extended the formal model into fuzzy domain and discussed topological relations. Clementini and Di Felice (1996, 1997) utilised the 9-intersection for the identification of fuzzy relations by introduction of a broad boundary algebraically. Based on the RCC theory, the egg-yolk model was proposed and 46 relations are identified for two fuzzy regions (Cohn and Gotts 1996). Zhan (1997) discussed the topological relations based on α-cut levels of fuzzy regions. Winter (2000) gave us an example how to calculate uncertainty of topological relations between two imprecise fuzzy regions.

The two models (Clementini and Di Felice 1996, and Cohn and Gotts 1996), among these researches, describe the topological relations between two fuzzy regions intensively. The Egg-yolk model, introduced by Cohn and Gotts (1996), are based on RCC (Region connection calculus) theory (Randell et al 1992). Eight base relations can be defined where one and only one will hold between a given pair of regions. The Egg-yolk model is an extension of RCC theory into the fuzzy domain. The *egg* is the maximal extent of a vague region and the *yolk* is its minimal extent, while the *white* is the area of indeterminacy. 46 relations are identified based on the so-called limits on the possible 'complete crispings' or precise versions of a vague region. Any acceptable complete crisping must lie between the inner and outer limits of defined yolk and egg.

Clementini and Di Felice (1996) proposed an algebraic model for topological relations between fuzzy objects, based on the 9-intersection approach. *A region with a broad boundary* is made up of two simple regions. The broad boundary is defined by a closed connected subset of R^2 with a hole. By use of the 9-intersection approach, 44 different relations are identified.

By comparison of the egg-yolk model with the algebraic model, it can be found that two relations, which appear in the egg-yolk model, are missing in the algebraic model. On the other hand, some relations in the algebraic model cannot be identified by the Egg-yolk model. Fundamentally the algebraic model, which is based on geometric, discusses the topological relations in the *crisp* topological space. In such a space, every subset should be crisp. Therefore, the broad boundary can be regarded as a closed set in the crisp topological space, which is projected from a fuzzy set.

Another way to interpret the topological relations between two fuzzy regions is to put them in the fuzzy topological space, the extension of the crisp topological space, since the fuzzy topological space can accommodate fuzzy sets naturally. Tang and Kainz (2001) investigated the topological relations between two fuzzy regions in a special fuzzy topological space. In their defined fuzzy topological space, the boundary of a fuzzy region can be decomposed into the boundary of the boundary and the interior of the boundary. 3*3, 4*4 and 5*5 intersection matrices between two fuzzy sets can be formalized. They verified that there are 44 relations in the crisp fuzzy topological space, and also identified 152 relations between two fuzzy regions based on the extended 4*4 intersection approach.

This paper deals with the topological relations between fuzzy regions in a general fuzzy topological space. For the remainder of this paper, Section 2 introduces the notations of fuzzy boundary and discusses topology. Section 3 discusses the mutually disjoint conditions between the interior, boundary and the exterior in the fuzzy topological space. Section 4 proposes some novel properties of a fuzzy topological space and formalizes a 9-intersection matrix and a 4*4 intersection matrix. Section 5 defines a simple fuzzy region topologically based on the simple crisp region in the crisp topological space. Section 6 identifies the topological relations between two simple fuzzy regions. Section 7 shows a method for the extension of topological relations by use of the 4*4 intersection matrix.

2 Some notations of fuzzy topological space

1.1 Fuzzy topology

Fuzzy topology, as an extension of general (crisp) topology, has several definitions. The notations introduced here are based on the definition proposed by Chang (1968).

Supposing A is a fuzzy subset of an ordinary set X, $[0,1]^X$ is the fuzzy power set of X. $\forall \delta \subseteq [0,1]^x$, if

$$\phi, X \in \delta;$$

$$\forall A_i \in \delta, \vee_I A_i \in \delta;$$

$$\forall U, V \in \delta, U \wedge V \in \delta$$

δ is called a *fuzzy topology on X ($i \in I$ is an index set)*. *(X, δ)* is called a *fuzzy topological space* for X. For simplicity, call it *fts on X* in short and denote it by *(X, δ)* or simply, *X*. Call the crisp topological space on *X cts on X* for short, and denote it by *(X, τ)*. Every element in δ is called an *open set* in *(X, δ)*. The *complement* A^c of an open subset A is called a *closed set*. Let A be a fuzzy set in *(X, δ)*, The union of all the open sets contained in A is the *interior of A*, denoted by A°. A° is the largest open set contained in A. The intersection of all the closed sets containing A is called the *closure of A*, denoted by A^{-}. A^{-} is the smallest closed set containing A. *The exterior of A* is the complement of A^{-} and denoted by A^{e}. Obviously A^{e} is an open set.

For later discussions, some standard results about the fts are listed in Proposition 2.1 (refer to Liu and Luo 1997).

Proposition 2.1

1. $A^{o} \wedge B^{o} = (A \wedge B)^{o}, A^{o} \vee B^{o} \leq (A \vee B)^{o}$
2. $A^{-} \wedge B^{-} \geq (A \wedge B)^{-}, A^{-} \vee B^{-} = (A \vee B)^{-}$
3. $A^{o} \leq A \leq A^{-}$
4. $A^{oc} = A^{c-}, A^{-c} = A^{co}$

A fuzzy set A is said to be *regular open* iff $A = A^{-o}$; it is *regular closed* iff $A = A^{o-}$.

Proposition 2.2: If A is regular closed, then A° is regular open. If A is regular open, then A⁻ is regular closed.

Proof: If $A = A^{o-}$, then $A^{o} = (A^{o-})^{o} = A^{o-o}$. If $A = A^{-o}$, then $A^{-} = (A^{-o})^{-} = A^{-o-}$.

Proposition 2.3: If A and B are regular open, then A∧B is regular open.

Proof: $A \wedge B = A^{-o} \wedge B^{-o} = (A^{-} \wedge B^{-})^{o} \geq (A \wedge B)^{-o} \geq (A \wedge B)^{o} = A^{o} \wedge B^{o} = A \wedge B$.

2.2 Fuzzy point, neighbourhood, and quasi-neighbourhood

A fuzzy set in X is called *a fuzzy point* iff it takes the value 0 for all $y \in X$, except one, say $x \in X$ (Wong 1974, Pu and Liu 1980). We denote the fuzzy point by x_{λ} $(0<\lambda\leq 1)$, *i.e.,* the value at x is λ, and call the point x *its support*. The fuzzy point x_{λ} is contained in a fuzzy set A or belongs to A, denoted by $x_{\lambda} \in A$ iff $\lambda \leq A(x)$, where $A(x)$ is the membership degree of x to A.

A fuzzy set A in (X, δ) is called *a neighbourhood* of fuzzy point x_{λ} iff there exists a $B \in \delta$ such that $x_{\lambda} \in B \leq A$. A fuzzy point $x_{\lambda} \in A^{\circ}$ iff x_{λ} has a neighbourhood contained in A. Obviously, a fuzzy point $x_{\lambda} \notin A^{\circ}$ iff every neighbourhood of x_{λ} is not contained in A.

A fuzzy point x_{λ} is called *quasi-coincident with A* denoted by $x_{\lambda} \hat{q} A$, iff $\lambda > A^{c}(x)$ or $\lambda + A(x) > 1$. Call *fuzzy set A quasi-coincident with B* iff $A(x) > B^{c}(x)$ or $A(x)+B(x)>1$. A fuzzy set A in (X, δ) is called a *quasi-neighbourhood of* x_{λ} iff there exists a $B \in \delta$ such that $x_{\lambda} \hat{q} B \leq A$. A fuzzy point $x_{\lambda} \in A^{-}$ iff each quasi-neighbourhood of x_{λ} is quasi-coincident with A.

For more discussions, we define *pan-neighbourhood P* of fuzzy point x_{λ} iff there exists a $B \in \delta$ such that $B \leq P$ and $x_{\lambda} \wedge B(x) \neq \phi$. Obviously $A(x)=A^{-}(x)$ iff there is a pan-neighbourhood $P \in \delta$ of x such that $A(x)+P(x)=1$. $A(x)=A^{\circ}(x)$ iff there is a P of x such that $A(x)=P(x)$.

2.3 Separation and connectedness

Two fuzzy sets A and B in (X, δ) are said to be *separated* iff there exist U, $V \in \delta$ such that $U \geq A$, $V \geq B$ and $U \wedge B = V \wedge A = \phi$. Two fuzzy sets A and B in (X, δ) are said to be *Q-separated* iff there exist closed sets H, K such that $H \geq A$, $K \geq B$ and $H \wedge B = K \wedge A = \phi$. The Q-separation and separation do not imply each other. If A and B are Q-separated, they cannot be separated (Pu and Liu 1980).

The concept of connectedness of a fuzzy set is still under development. We define the connectedness based on the separation and Q-separation. A fuzzy set A is said to be *open connected* if there is no separated C and D such that $A=C \vee D$. A fuzzy set A is said to be *closed connected* if there is no Q-separated C and D such that $A=C \vee D$. A fuzzy set is said to be *double connected* if it is both open connected and closed connected.

2.4 Fuzzy mapping and homeomorphism

Let $f:X \to Y$ an ordinary mapping, $(X,\delta),(Y,\mu)$ be fts, the *fuzzy mapping* and its *reverse mapping* are defined by (Chang 1968):

$$f^{\to} : (X,\delta) \to (Y,\mu), f^{\to}(A)(y) = \vee\{ A(x) : x \in X, f(x) = y \}, A \in [0,1]^X, y \in Y$$

$$f^{\leftarrow} : (Y,\mu) \to (X,\delta), f^{\leftarrow}(B)(x) = B(f(x)), B \in [0,1]^Y, x \in X$$

Call $f^{\to}$ *continuous* if its reverse mapping $f^{\leftarrow}:(Y,\mu) \to (X,\delta)$ maps every open subset in (Y,μ) as an open subset in (X,δ). Call it *open* if it maps every open subset in (X,δ) as an open subset in (Y,μ). $f^{\to}$ is called *fuzzy homeomorphism* if it is bijective, continuous and open. Fuzzy homeomorphism is *union preserving* and *crisp preserving* (Liu and Luo 1997). The properties of a fuzzy set that is invariant under fuzzy homeomorphisms are *topological properties*.

2.5 Fuzzy topological relation

Suppose on the product space $U \times V$, there is a mapping R: $U \times V \to [0,1]$. Its membership function μ_R decides the relation degree of ordered pair (u,v). Then R is a fuzzy relation from U to V. If $\mu_R(u,v)$ takes only values 0 and 1, then R is a crisp relation. If $U=V$, then R is called a binary fuzzy relation on U.

A fuzzy relation R between A and B on the fts U is called a fuzzy topological relation if R is preserved under a fuzzy homeomorphism of its embedding fts.

2.6 The 4-intersection and 9-intersection approaches

The 4-intersection and the 9-intersection approaches are well-known methods for identification of (crisp) topological relations between two crisp sets in the cts. In the cts, the following propositions exist for a set A:

Proposition 2.4 $A^- = \partial A \cup A^o$; $\partial A \cap A^o = \phi$

Between the interior and boundary, and between the interior, boundary and exterior of set A and B there are four and nine intersections respectively: $A^o \cap B^o$, $A^o \cap \partial B$, $\partial A \cap B^o$, $\partial A \cap \partial B$, and $A^o \cap B^o$, $A^o \cap \partial B$, $A^o \cap B^e$, $\partial A \cap B^o$, $\partial A \cap \partial B$, $\partial A \cap B^e$, $A^e \cap B^o$, $A^e \cap \partial B$, $A^e \cap B^e$. The *empty/non-empty contents* of these intersections are topological invariants. Therefore, the 4-intersection and the 9-intersection can be used for the description of topological relations between A and B.

3 Fuzzy boundary

3.1 Definitions of fuzzy boundary

The definition of fuzzy boundary has several extensions in the fts:

1. Warren (1977): The *fuzzy boundary* of a fuzzy set A is an infimum of all closed fuzzy sets D in X with the property $D(x) \geq A^-(x)$ for all $x \in X$ for which $A^- \wedge A^{c-}(x) > 0$.
2. Pu and Liu (1980): The *fuzzy boundary* of a fuzzy set A is the intersection of the closure with the closure of the complement of a fuzzy set, i.e., $\partial A_I = A^- \wedge A^{c-}$. This form is identical to the boundary defined in the cts.
3. Wu and Zheng (1991) defined a *lattice-fuzzy boundary* in the lattice-fuzzy topological space. The fuzzy boundary is the union of the molecules that are less than the closure and not less than the interior of a set. Simplifying it into a fts, the *fuzzy boundary* of a fuzzy set is $\partial A = \vee\{ x_\lambda : \lambda \leq A^-(x) \text{ and } \lambda > A^o(x)\}$.
4. Cuchillo-Ibanez and Tarres (1997): The *fuzzy boundary* of a fuzzy set A is the infimum of all closed fuzzy sets D in X with the property $D(x) \geq A^-(x)$ for all $x \in X$ for which $A^-(x) > A^o(x)$.

3.2 Fuzzy boundary and related properties

In the cts, the boundary *(∂A)* of a subset *A* is the intersection of the closure of *A* with the closure of the complement of *A*, i.e., $\partial A = A^- \cap A^{c-}$. To keep the boundary of a set consistent in both fts and cts, we adopt the definition of Pu and Liu.

Definition 3.1 The fuzzy boundary of a fuzzy set is defined as the intersection of the closure and the closure of the complement of a fuzzy set in the fts: $\partial A = A^- \wedge A^{c-}$.

Some properties related to the boundary of a fuzzy set in the fts are listed in Table 1. For comparison, the boundary of a subset in the cts is also shown.

Table 1. . Some properties related to the boundary in different topological spaces

Topological spaces / Boundaries and their properties	Ordinary (crisp) topological space (X,τ)	Fuzzy topological space (X,δ)
Definitions of the boundary of a subset $\widetilde{A}$ (A)	$\partial A = A^- \cap A^{c-}$	$\partial\widetilde{A} = \widetilde{A}^- \wedge \widetilde{A}^{c-}$
The boundary and the boundary of the boundary	$\partial(\partial A) \subseteq \partial A$	$\partial(\partial\widetilde{A}) \leq \partial\widetilde{A}$
Decomposition of the boundary	$\partial A = (\partial A)^o \cup \partial(\partial A)$	$\partial\widetilde{A} = (\partial\widetilde{A})^o \vee \partial(\partial\widetilde{A})$
Property of the boundary	$A^- = A^o \vee \partial A$	$A^- \geq A^o \vee \partial A$
Property of the boundary	$\partial A = A^- - A^o$	$\partial\widetilde{A} \geq \widetilde{A}^- - \widetilde{A}^o$
The boundary of the boundary	$\partial(\partial A) = \partial(A^o) \cup \partial(A^-)$	--

3.3 Conditions for mutual disjointness

Our aim is to analyse the topological relations between two regions. The well-known 4- and 9-intersection approaches are the best ways to realize them. Proposition 2.4 indicates two basic conditions for adopting the 9-intersection approach, i.e., the topological parts, A^o, ∂A and A^e of a subset, should be mutually disjoint; and the closure is the union of the boundary with the interior.

However, in the fts the intersections between the interior, boundary, and exterior may be not disjoint. For example, the intersection between ∂A and A^o can not hold in the general fts. There could be $\partial A > A^o > \phi$ when $A^- > A^{oc} > \phi$. About the conditions for these three parts to be mutually disjoint, we have the following propositions.

Proposition 3.3 $\partial A \wedge A^o = \phi$ *iff* A^o *is crisp.*

Proof: $\Rightarrow$ $\forall x \in X$ if $0 < A^o(x) < 1$, then $0 < A^-(x) \leq 1$ and $0 < A^{oc} < 1$. Then $(\partial A \wedge A^o)(x) = (A^- \wedge A^{c-} \wedge A^o)(x) = (A^- \wedge A^{oc} \wedge A^o)(x) \neq 0$. So if $(\partial A \wedge A^o)(x) = 0$ then $A^o(x) = 0$ or $A^o(x) = 1$.

$\Leftarrow$ Assume A^o is crisp. Then $A^o(x) = 0$ or $A^o(x) = 1$. If $A^o(x) = 1$, then $(\partial A \wedge A^o)(x) = 0$. If $A^o(x) = 1$, then $A^-(x) = 1$, so $A^{-c}(x) = 0$. Then $(A^{c-} \wedge A^o)(x) = 0$. So $(A^- \wedge A^{c-} \wedge A^o)(x) = (\partial A \wedge A^o)(x) = 0$.

This proposition shows that, in order to make the boundary and the interior disjoint in the fts, the interior should be crisp.

Proposition 3.4 $A^- \wedge A^e = \phi$ *iff* A^- *and* A^e *is crisp.*

Proof: $\Leftarrow$ is clear.

$\Rightarrow$ If $(A^- \wedge A^e)(x) = 0$, then either $A^-(x) = 0$ or $A^e(x) = 0$. If $A^e(x) = 0$, then $A^-(x) = 1$. If $A^-(x) = 0$, then $A^e(x) = 1$. So A^- and A^e are crisp.

In other words, the sufficient and necessary condition for the closure and the exterior to be disjoint is that both the closure and the exterior are crisp sets.

Proposition 3.5 *If A^- is crisp, then $\partial A \wedge A^o = \phi$ iff ∂A and A^o are crisp.*

Proof: $\Leftarrow$ is clear.

$\Rightarrow$ According to Proposition 3.2, $\partial A \wedge A^o = \phi$ iff A^o is crisp. If A^- is also crisp, then ∂A is crisp. So the proposition holds.

When the closure is crisp, in order to make the boundary and the interior disjoint, both the interior and the boundary should be crisp.

The above propositions reveal that to utilize the 9-intersection matrix, the interior and boundary of a fuzzy set should be crisp. Tang and Kainz (2001) proposed a special fts in which the mutual disjointness of the interior, boundary and exterior of a subset hold. Although Definition 3.2 is different with Tang and Kainz's boundary, the above propositions proved that if all open sets are crisp then the interior, the boundary and the exterior are mutually disjoint. Therefore if we apply this definition of a fuzzy region in their defined crisp fts, and take the empty/non-empty contents in the 9-intersections or 4*4 intersections as topological invariants, we can also deliver 44 or 152 kinds of the topological relations between two fuzzy regions.

4 Some topological notations of fts and intersection matrices

4.1 Definitions

Section 3 discussed some properties related to the fuzzy boundary. It has been shown that the proposition 2.4 cannot hold in the general fts. Tang and Kainz's method for topological relations is to generate a special fts in which the topological parts (the interior, boundary and exterior of a subset) are mutually disjoint. Now we propose a new method to form the intersection matrix.

In the fts, the closure and the boundary of a fuzzy set can be decomposed based on relations of the elements of these sets.

Definition 4.1 The boundary of fuzzy set A in the fts can be decomposed into two subsets in terms of the closure of fuzzy set A: call the subsets of the boundary ∂A where $\partial A(x) = A^-(x)$ and $\partial A(x) < A^-(x)$ the c-boundary and the i-boundary of fuzzy set A, respectively. Denote them by $\partial^c A$ and $\partial^i A$. That is $\partial^c A = \{x_\lambda \in \partial A : \partial A(x) = A^-(x)\}$; $\partial^i A = \{x_\lambda \in \partial A : \partial A(x) < A^-(x)\}$. Obviously $\partial^c A = \phi$ iff $A^-(x) > A^{c-}(x) > 0$ for $x \in X$. $\partial^c A = \partial A$ iff $0 < A^-(x) \le A^{c-}(x)$.

Definition 4.2 The closure of fuzzy set A in the fts can be decomposed into two subsets in terms of ∂A: call the subsets of the closure A^- where $\partial A(x) < A^-(x)$ and $\partial A(x) = A^-(x)$ the i-closure and c-closure of A and denote them by $A^\pm$ and $A^\mp$. That is $A^\pm = \{x_\lambda \in A^- : \partial A(x) < A^-(x)\}$; $A^\mp = \{x_\lambda \in A^- : \partial A(x) = A^-(x)\}$.

Proposition 4.3 (1) $\partial^c A = A^\mp$

(2) $\partial^c A \wedge A^\pm = \phi$

(3) $A^- = \partial^c A \vee A^\pm$

Proof: obvious according to the definitions.

Definition 4.4 Define the subset of the interior of fuzzy set A in the fts where $\forall x \in X$, $A^o(x) = 1$ by the core of A, and denote it by $A^\oplus$. Obviously $A^\oplus$ is the crisp subset of A^o. Define the outer $A^=$ of fuzzy set A by the fuzzy points x_λ (λ>0): $A^= = \{x_\lambda \in A^e : A^e(x) = 1\}$. Obviously $A^=$ is the crisp subset of A^e, and $A^= \wedge A^- = \phi$.

The following proposition holds between $A^\pm$ and $A^\oplus$:

Proposition 4.5 $A^{\pm} \geq A^{\oplus}$

Proof: if $\forall x_\lambda \in X, A^{\pm}(x) = A^{-}(x)$ iff $\forall x_\lambda \in X, \partial A(x) < A^{-}(x)$. When $A^{o}(x) = 1$, then $A^{oc}(x) = 0$, then $(A^{-} \wedge A^{oc})(x) = (A^{-} \wedge A^{c-})(x) = 0$. Then $\partial A(x) = 0$. Since $A^{-}(x) = A^{o}(x) = 1$. So $A^{\pm}(x) = 1$. That is, when $A^{o}(x) = 1$, then $A^{\pm}(x) = 1$. Therefore $A^{\pm} \geq A^{\oplus}$.

Definition 4.6 Define the b-closure of A in the fts by $A^{\pm} - A^{\oplus}$, and denote it by $A^{\perp}$, i.e., $A^{\perp} = A^{\pm} - A^{\oplus}$. Define the join of the c-boundary with the b-closure of fuzzy set A by the fringe of A and denote it by ℓA, i.e., $\ell A = \partial^{c} A \vee A^{\perp}$.

Proposition 4.7

(1) If $A^{\pm}$ is not empty, then $\forall x_\lambda \in A^{\pm}$, $A^{\pm}(x) > 0.5$

(2) If $(\partial^{c} A)^{o} \neq \phi$, then $\forall x_\lambda \in (\partial^{c} A)^{o}$, $\forall x_\lambda \in (\partial^{c} A)^{o} \leq 0.5$

Proof: (1) $A^{\pm}(x) = A^{-}(x)$ iff $A^{-}(x) > \partial A(x)$ $\forall x \in X$. Then $A^{-}(x) > \partial A(x) = A^{-}(x) \wedge A^{c-}(x) > A^{c-}(x) = A^{oc}(x)$. On the other hand, $A^{-}(x) \geq A^{o}(x)$. Hence $A^{\pm}(x) = A^{-}(x) > 0.5$. Otherwise, if $A^{-}(x) \leq 0.5$, then $A^{-}(x) !> A^{oc}(x)$. Therefore $A^{\pm}(x) > 0.5$.
(2) Refer to (1).

Therefore, it is reasonable to call the b-closure by *the major fringe*, and the c-boundary by *the minor fringe of fuzzy set A*.

Through above definitions the closure of fuzzy set *A* is decomposed into the core, the b-closure and the c-boundary. According to their definitions, the following proposition holds.

Proposition 4.8

1. $A^{-} = A^{\oplus} \vee A^{\perp} \vee \partial^{c} A = A^{\oplus} \vee \ell A$
2. $A^{\oplus}, A^{\perp}$ *and* $\partial^{c} A$ are mutually disjoint.
3. $A^{\oplus}$ *and* ℓA are mutually disjoint.

4.2 Topological properties

The details of these notations are neglected since they are not relevant to the paper. We will show that although the c-boundary, i-boundary, c-closure, i-closure, b-closure, core, and outer are subsets of topological notations, it can be proven that they are topological properties of the fts.

Proposition 4.9 A fuzzy homeomorphism is boundary-preserving, i.e., if $f^{\rightarrow}$ is a homeomorphism, then $f^{\rightarrow}(\partial A) = \partial f^{\rightarrow}(A)$.

Proof: According to Wu and Zheng, if $f^{\rightarrow}$ is a homeomorphism induced by $f(X) = Y$, where f is an ordinary mapping, then $f^{\rightarrow}$ *i*s closure-preserving, that is $f^{\rightarrow}(A^{-}) = (f^{\rightarrow}(A))^{-}$. Supposing $f^{\rightarrow}(A) = B$, then $f^{\rightarrow}(\partial A) = f^{\rightarrow}(A^{-} \wedge A^{c-}) =$ $f^{\rightarrow}(A^{-}) \wedge f^{\rightarrow}(A^{c-}) =$ $(f^{\rightarrow}(A))^{-} \wedge (f^{\rightarrow}(A^{c}))^{-} = B^{-} \wedge B^{c-} = \partial B$.

Proposition 4.10 The c-boundary, i-boundary, c-closure, i-closure, b-closure, core, and outer of a fuzzy set *A are topological properties.*

Proof: Supposing $f^{\rightarrow}$ is a homeomorphic mapping from X to Y: $f^{\rightarrow}(A) = B$, then $f^{\rightarrow}(A^{-}) = (f^{\rightarrow}(A))^{-}$ and $f^{\rightarrow}(\partial A) = \partial f^{\rightarrow}(A)$. That is $f^{\rightarrow}(A(x)^{-}) = (f^{\rightarrow}(A))^{-}(x) = B^{-}(y)$ and $f^{\rightarrow}(\partial A(x)^{-}) = (\partial^{\rightarrow} f(A))(x) = \partial B(y)$. Then if $(A(x))^{-} = (\partial A)(x)$ then $(f^{\rightarrow}(A(x))^{-} = f^{\rightarrow}(\partial A(x))$, which is $B(y)^{-} = \partial B(y)$, therefore $f(\partial^{c} A) = \partial^{c} B$. So c-boundary is a topological property. Similarly, the i-boundary, c-closure and i-closure are topological properties.

The core is a topological property since it is a crisp subset of A^{o}. The outer is a crisp subset of A^{-c}, therefore it is a topological property. The b-closure $A^{\perp}$ is a topological property since $A^{\perp} = A^{\pm} - A^{\oplus} = A^{\pm} \wedge A^{\oplus c}$.

4.3 A 9-intersection matrix and a 4*4-intersection matrix

Propositions 4.8 shows that the core, b-closure, c-boundary and the outer of a fuzzy set *A* are mutually disjoint. Proposition 4.10 reveals that they are topological parts. We call the core, b-closure, c-boundary and the outer by the topological parts of fuzzy set *A*. Likewise, the core, the fringe and the outer can be called by the topological parts of fuzzy set *A*. In the cts, such definitions are superfluous since the interior, boundary and exterior meet the conditions of Proposition 2.4. In the fts, we can adopt the above notations to replace the topological parts in the 9-intersection matrix formalized in the cts.

Supposing there are two fuzzy sets *A* and *B* in the fts, we adopt the core, fringe and outer as three topological parts to formalize the 9-intersection matrix (the reason will be explained later). Between two closed sets *A* and *B, the 9-intersection matrix* will be:

$$I_9 = \begin{bmatrix} A^{\oplus} \wedge B^{\oplus} & A^{\oplus} \wedge \ell B & A^{\oplus} \wedge B^{=} \\ \ell A \wedge B^{\oplus} & \ell A \wedge \ell B & \ell A \wedge B^{=} \\ A^{=} \wedge B^{\oplus} & A^{=} \wedge \ell B & A^{=} \wedge B^{=} \end{bmatrix}$$

If we decompose the closure of a fuzzy set into the core, b-closure, c-boundary and outer, then between two closed sets A and B, the *4*4 intersection matrix* can be formalized:

$$I_{4*4} = \begin{bmatrix} A^{\oplus} \wedge B^{\oplus} & A^{\oplus} \wedge B^{\perp} & A^{\oplus} \wedge \partial^c B & A^{\oplus} \wedge B^{=} \\ A^{\perp} \wedge B^{\oplus} & A^{\perp} \wedge B^{\perp} & A^{\perp} \wedge \partial^c B & A^{\perp} \wedge B^{=} \\ \partial^c A \wedge B^{\oplus} & \partial^c A \wedge B^{\perp} & \partial^c A \wedge \partial^c B & \partial^c A \wedge B^{=} \\ A^{=} \wedge B^{\oplus} & A^{=} \wedge B^{\perp} & A^{=} \wedge \partial^c B & A^{=} \wedge B^{=} \end{bmatrix}$$

The topological relations can be described by the topological invariants of the intersection sets in the fts. To any intersection, the empty/non-empty contents are still topological invariants of the fts. In general there are 512 topological relations between two fuzzy sets if the 9-intersection matrix is adopted. 2^{16}= 65536 relations can be derived between two fuzzy sets by use of the 4*4 intersection matrix. For GIS applications, we would impose some conditions to a fuzzy set, which can be regarded as topological definition of a fuzzy region.

5. A simple fuzzy region

5.1 Definition of a simple crisp region

Traditionally, a simple crisp region is defined in the cts. Correspondingly, a fuzzy region should be defined in the fts. A "*real"* fuzzy region (where the membership values at some points are between 0 and 1) cannot be accommodated in the cts since every cts contains only crisp sets. On the other hand, a simple crisp region can be accommodated by the fts since a crisp region is just a special form of a fuzzy region. Therefore, when we define a fuzzy region, we should consider its special form - a simple crisp region - the consistency in both crisp and fuzzy topological spaces. In the cts, a simple fuzzy region is defined as a regular closed set with a connected interior. We adopt this definition for the simple crisp region in the fts.

Definition 5.1 A subset is called a simple crisp region in the fts if it meets the following conditions:
1. It is a non-empty proper double connected regular crisp set;
2. Its interior is a non-empty proper double connected open crisp set.

In the cts, the connectedness of a set means that the set cannot be separated by two disjoint closed sets or disjoint open sets. However, in the fts Q-separation and separation do not imply each other (see Section 2). Therefore we adopt closed connectedness and open connectedness for a simple crisp region in the fts.

5.2 Definition of a simple fuzzy region

In Definition 5.1 the interior and the boundary of a simple crisp region are crisp. To preserve such a property for a simple crisp region in the fts, we can require the core, which is a crisp subset of the interior of a fuzzy set, be an open set of the fts, and require it to be the interior of a simple crisp region. Therefore we define a *simple fuzzy region* if it meets the following conditions.

Definition 5.2 A subset is called a simple fuzzy region in the fts if it meets the following conditions:
1. A simple fuzzy region is a non-empty double connected regular closed set;
2. The core is the interior of a simple crisp region;
3. The i-closure is a non-empty double connected regular open set;
4. The interior of the b-closure is a non-empty double connected regular open set;
5. The c-boundary is a non-empty double connected closed set;
6. The outer is a non-empty double connected open set.

The first condition is similar to the definition of a simple crisp region in the cts with the exception that the connectedness is replaced by the double connectedness. In the second condition, the core is defined as an open set, which is similar to the interior of a simple crisp region in the cts. (In the cts the element of the non-empty interior of a subset is equal to 1). This is also the reason why we adopt the core, fringe and outer of a fuzzy set as the topological parts in the fts. The third condition also is an extension of the interior of a subset from the cts to the fts. (because the subsets of the closure which is greater than the boundary of a crisp region in the cts is regular open, it is appropriate to define such a subset in the fts also to be regular open). The fourth condition ensures that there are some real fuzzy points belonging to the i-closure. It is also reasonable to define the c-boundary to be closed. Figure 1 shows such a definition in R^2.

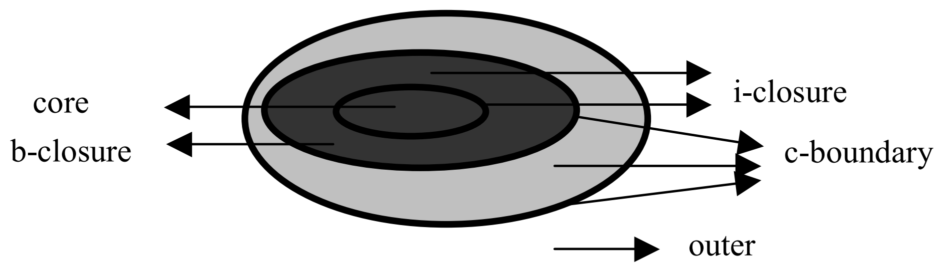

Fig. 1. A simple fuzzy region

5.3 Some properties of a simple fuzzy region

Proposition 5.3 If A is a simple fuzzy region, then the fringe is regular closed.
Proof: According to the definition of ℓA, $\ell A = A^- \wedge A^{\oplus c}$. Then ℓA is closed. $(\ell A)^o = (A \wedge A^{\oplus -})^o = A^o \wedge A^{\oplus co}$. Since $A^{\oplus co}$ and A^o are regular open, then $(\ell A)^o$ is regular open. Thus *the fringe is regular closed.*

Proposition 5.4 The interior of the fringe of the core of a fuzzy set is empty, i.e., $(l(A^{\oplus}))^o = \phi$.
Proof: According to the definition of $A^{\oplus}$, $A^{\oplus} \wedge \ell(A^{\oplus}) = \phi$. If $(\ell(A^{\oplus}))^o \neq \phi$, then $A^{\oplus o} \vee (\ell(A^{\oplus}))^o > A^{\oplus o}$, then $A^{\oplus}$ is not regular open.

6. Topological relations between two simple regions

The topological relations between two fuzzy regions can be identified by use of the 9-intersection matrix. Since a simple fuzzy region is not a general fuzzy set, not all 512 relations can be realized in the 9-intersection matrix. Between two fuzzy regions, there are some limitations. For example, we can prove that the core is always “surrounded” by the fringe of a fuzzy set, which is similar to the interior being encompassed by the boundary of a simple crisp region in the cts.

6.1 Some limitations between two simple regions

Proposition 6.1 If A's core does not intersect with B's core, then A's core does not intersect with the fringe of B's core, and B's core does not intersect with the fringe of A's core, i.e., if $A^{\oplus} \wedge B^{\oplus} = \phi$, then $A^{\oplus} \wedge \ell(B^{\oplus}) = \phi$ and $B^{\oplus} \wedge \ell(A^{\oplus}) = \phi$.

Proof: Suppose *B*'s core intersects with the fringe of $A^{\oplus}$, that is $B^{\oplus} \wedge \ell(A^{\oplus}) = D$, then $D^{\circ}=\phi$ according to Proposition 5.4. That is $\forall x \in X$, $D^{\circ}(x)=0$. Then $\forall x_{\lambda} \in D$, all pan-neighbourhoods of x_{λ} intersect with $A^{\oplus}$ at x_{λ}, as well as *y* where $supp(y) \neq supp(x_{\lambda})$. On the other hand, since $D \leq B^{\oplus}$, $B^{\oplus}$ is a pan-neighbourhood of x_{λ}. So $A^{\oplus} \wedge B^{\oplus} \neq \phi$. Contradiction! The other part of the proposition can be proven in the same way.

Proposition 6.2 If A's core intersects with the fringe of B' core, then it must intersect with B's core, i.e., if $A^{\oplus} \wedge \ell(B^{\oplus}) \neq \phi$, then $A^{\oplus} \wedge B^{\oplus} \neq \phi$.

Proof: Suppose *A*'s fringe cannot intersect with *B*'s fringe, then *A*'s fringe cannot intersect with the fringe of B's core according to Proposition 6.1. Contradiction!

Proposition 6.3 If A's core intersects with B's core and outer, then it must also intersect with B's fringe, i.e., if $A^{\oplus} \wedge B^{\oplus} \neq \phi$ and $A^{\oplus} \wedge B^{=} \neq \phi$ then $A^{\oplus} \wedge \ell B \neq \phi$.

Proof: We will show that B's core is disjoint with B's outer. Suppose $A^{\oplus} \wedge B^{\oplus} = C \neq \phi$, and $A^{\oplus} \wedge B^{=} = D \neq \phi$, but $A^{\oplus} \wedge \ell(B^{\oplus}) = \phi$. Then $C \vee D = A^{\oplus}$ since C is crisp and $B^{=}$ is also crisp. Since C and D are open, therefore $A^{\oplus}$ is not open-connected. Contradiction!

Proposition 6.4 If A's core is disjoint with B's core, and A's fringe intersects with B's core, then A's fringe intersects with B's fringe, if $A^{\oplus} \wedge B^{\oplus} = \phi$, and $\ell A \wedge B^{\oplus} \neq \phi$ then $\ell A \wedge \ell B \neq \phi$.

Proof: Suppose $A^{\oplus} \wedge B^{\oplus} = \phi$, and $\ell A \wedge B^{\oplus} \neq \phi$ but $\ell A \wedge \ell B = \phi$. Then $B^{\oplus} \wedge \ell(A^{\oplus}) = \phi$ according to Proposition 6.1, and then $A \wedge \ell(B^{\oplus}) = \phi$. Then $A \wedge B^{\oplus} = \phi$. So $\ell A \wedge B^{\oplus} = \phi$. Contradiction.

Proposition 6.5 If A's core and fringe intersect with B's fringe, then the fringe of A's core intersects with B's fringe: if $A^{\oplus} \wedge \ell B \neq \phi$ and $\ell A \wedge \ell B \neq \phi$, then $\ell B \wedge \ell(A^{\oplus}) \neq \phi$.

Proof: Let $A^{\oplus} \wedge \ell B \neq \phi$, $\ell A \wedge \ell B \neq \phi$. Suppose $\ell B \wedge \ell(A^{\oplus}) = \phi$, then $\ell B \wedge A^{\oplus -} = C$ is closed, $\ell B \wedge A^{\oplus c} = D$ is closed. Then $\ell B = C \vee D$. Contradiction!

6.2 Topological relations by use of the 9-intersection matrix

Topological relations between two simple fuzzy regions can be identified by use of the 9-intersection matrix in the fts. We will identify the topological relations between two fuzzy regions in the fuzzy R^2. The simple fuzzy region is further limited as a two-dimensional and bounded set. The following conditions, which hold between two regions with broad boundary in the cts (referring to Clementini and Di Felice 1996), also hold in the fts according to the above propositions. The only difference lies in that the topological parts are changed into the core, fringe and outer of a simple fuzzy region.

1. The outers of two fuzzy regions intersect with each other;
2. Any part of one fuzzy region must intersect with at least one part of the other fuzzy region, and vice versa;
3. If one fuzzy region's core intersects with the other's core and outer, then it must also intersect with the other's fringe, and vice versa;
4. If both cores are disjoint, then one fuzzy region's core intersects with the other's fringe, or with the other's outer, and vice versa;
5. If both cores are disjoint and one fuzzy region's fringe intersects with the other's core, then the two fringes must intersect with each other, and vice versa;
6. If one fuzzy region's core intersects with the other's outer, then its fringe must also intersect with the other's outer, and vice versa;
7. If one fuzzy region's core is a subset of the other fuzzy region, then its fringe must intersect with the other, and vice versa;

8. If one fuzzy region's core does not intersect with the other fuzzy region, then its core must intersect with the other's outer, and vice versa;
9. If both cores do not intersect with each other, then at least one fringe must intersect with its opposite outer;
10. If both fringes intersect with the opposite cores, then the fringes must also intersect with each other;
11. If one fuzzy region's fringe intersects with the other's core and outer, then it must also intersect with the other's fringe, and vice versa;
12. If one fuzzy region is a subset of the core of the other, then its outer must intersect with the other's core, and vice versa;

Under such conditions, we can derive 44 relations between two regions in the general fts (see Appendix). Although the number is the same as Tang and Kainz's as well as Clementini and De Felice's, the differences are that (1) the core is adopted in the 9-intersection matrix instead of the interior; (2) the 9-intersection matrix is built in the general fts.

7 The 4*4 intersection matrix and its significance

In the 9-intersection matrix, the fringe of a simple fuzzy region, as a closed set, is the join of the b-closure and c-boundary. If we adopt the four topological parts in the 4*4 intersection matrix, the identification of the topological relations will be difficult since the b-closure could be neither open nor closed. For the convenience of the identification, the following four disjoint topological parts: the closure of the core, the interior of the b-closure, the c-boundary, and the outer, are built up for a simple fuzzy region based on the following proposition.

Proposition 7.1

1. (1) $A^{\pm} = A^{\perp o} \vee A^{\oplus -}$
2. (2) $A^{\perp o} \wedge A^{\oplus -} = \phi$

Proof:

1. $A^{\perp o} \vee A^{\oplus -} = (A^{\pm} \wedge A^{\oplus c})^{o} \vee A^{\oplus -} = (A^{\pm o} \wedge A^{\oplus co}) \vee A^{\oplus -} = (A^{\pm o} \vee A^{\oplus -})$
$\wedge (A^{\oplus co} \vee A^{\oplus -}) = A^{\pm} \wedge (A^{\oplus - c} \vee A^{\oplus -}) = A^{\pm}$
2. $A^{\perp o} \wedge A^{\oplus -} = (A^{\pm} \wedge A^{\oplus c})^{o} \wedge A^{\oplus -} = A^{\pm o} \wedge A^{\oplus co} \wedge A^{\oplus -} = \phi$.

Therefore the 4*4 intersection matrix can be re-formalized as follows:

$$I_{4*4} = \begin{bmatrix} A^{\oplus -} \wedge B^{\oplus -} & A^{\oplus -} \wedge B^{\perp o} & A^{\oplus -} \wedge \partial^{c} B & A^{\oplus -} \wedge B^{=} \\ A^{\perp o} \wedge B^{\oplus -} & A^{\perp o} \wedge B^{\perp o} & A^{\perp o} \wedge \partial^{c} B & A^{\perp o} \wedge B^{=} \\ \partial^{c} A \wedge B^{\oplus -} & \partial^{c} A \wedge B^{\perp o} & \partial^{c} A \wedge \partial^{c} B & \partial^{c} A \wedge B^{=} \\ A^{=} \wedge B^{\oplus -} & A^{=} \wedge B^{\perp o} & A^{=} \wedge \partial^{c} B & A^{=} \wedge B^{=} \end{bmatrix}$$

The identification of the topological relations can be realized by decomposing the 44 relations. The core in the 9-intersection matrix can be replaced by the i-closure since they have similar properties for the identification. However, the number of topological relations between two simple fuzzy regions will exceed several hundreds. The list of all these possibilities is less meaningful. In GIS applications more concerns are taken on the relations between a specific spatial object with others. For example, what is the relation between a steep slope area and a piece of forest? What is the relation between a wetland and a swan habitat? According to Proposition 4.7, we can differentiate the main fringe with the minor fringe of a simple fuzzy region, which will provides more precise description for topological relations between two fuzzy regions. Figure 2 shows the refinement of a topological relation between a steep slope area and a piece of forest.

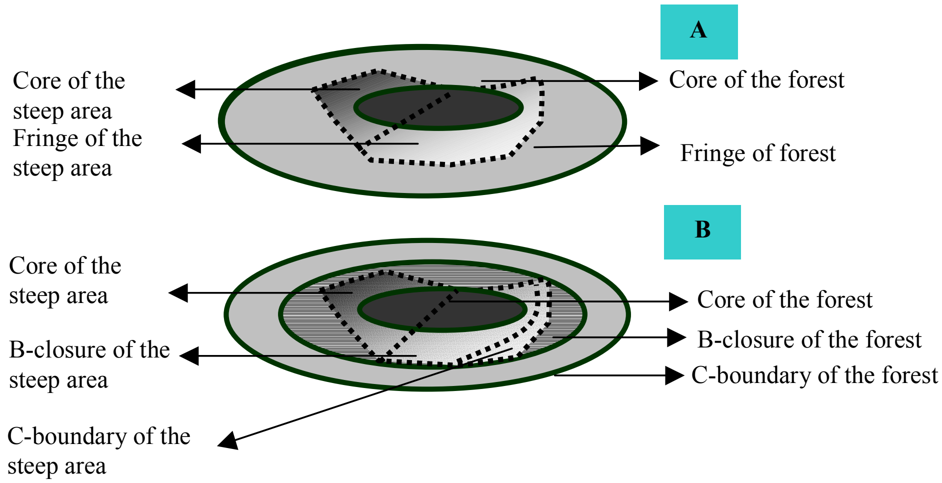

Fig. 2. . Comparison of topological relations by use of the 9-intersection matrix and the 4*4 intersection matrix. Figure A shows that the boundary and the core of the forest area contain the steep slope area. Figure B shows that the major boundary and the core of the forest area contains the steep slope area, which describes the topological relations more precisely.

8 Conclusion and discussion

Fuzzy objects exist in almost all natural phenomena. Almost all geometric objects, if considered as sets in a topological space, expose some degree of fuzziness. Every object has its boundary. The topological boundary of a subset, as one of the most important notations in the fts, is adopted as the starting point for analysis of topological relations between two fuzzy sets.

The main contributions of the paper are the formalism of the 9-intersection and 4*4 intersection matrices, and the identification of topological relations between two fuzzy regions in the *general* fts. For the formalism of intersection matrices, a structure of a closed set is decomposed into some novel notations including the core, b-closure and c-boundary, as well as into the core and fringe subsets. Although these notions are the subsets of the closure or the boundary, they are proved to be fuzzy topological properties of the fts. The 9-intersection matrix, and then the 4*4-intersection matrix can then be formalized. In the 9-intersection matrix, the topological parts are updated from the interior, the boundary and the exterior into the core, fringe and the outer, respectively. The core, b-closure, c-boundary and outer are also the topological parts of a fuzzy set.

The definition of the fuzzy simple region constitutes the premise for the topological relations between fuzzy spatial objects. This paper derives a simple fuzzy region based on the definition of a simple region in the crisp topological space. The definition of the simple crisp region, which inherits from a crisp region in the cts, is at first given in the fts. Then it is extended as a simple fuzzy region in the fts, in the way that it is a general form of the simple crisp region. Based on the 9-intersection matrix, 44 relations are identified. Although the number of the relations is the same as that of Clementini and Di Felice (1996) and Tang and Kainz (2001), it should be pointed out that these relations are formalised in the general fuzzy topological space.

An advantage about interpretation of topological relations in the general fts is that more details of the closure can be revealed, which cannot be derived in the cts or in some special fts. The separation of the major fringe with the minor fringe shows such advantages.

Due to the fuzziness of the regions, topological invariants other than empty/nonempty contents can be adopted for the identification of topological relations in the fts. For example, a non-empty intersection can be broken up into four comparisons, namely, $\geq, \leq, =, \leq\geq$ between the intersection parts of two fuzzy regions. These comparisons operators are also topological invariants of the fts, which cannot be achieved in the cts. These topological invariants will expand the topological relations between two regions from “horizontal” relations to “vertical” relations. We will discuss it in future publications.

This paper formalized some intersection matrices to investigate the topological relations between two simple fuzzy regions. These have to be extended for more complicated fuzzy regions. The topological relations are identified for 2-dimensional spatial objects. In the future we will also consider fuzzy spatial objects of different dimensions and will apply the results to image analysis in land cover change.

References

Allen J.F. (1983) Maintaining knowledge about temporal intervals. Communications of the ACM 26(11): 832-843

Al-Taha K.K.A. (1992) Temporal reasoning in cadastral systems. Ph.D. thesis, the Graduate School, University of Maine

Bennett B., Cohn A.G., Lsli A. (1997) A logical approach to incorporating qualitative spatial reasoning into GIS. In: Proceedings of the international conference on spatial information theory (COSIT) pp 503-504

Bennett B., Cohn A.G. (1999) Consistency of topological relations in the presence of convexity constraints. In: Proceedings of the hot topics in spatio-temporal reasoning workshop, IJCAI-99, Stockholm, Sweden

Chang C.L. (1968) Fuzzy topological space. Journal of Mathematical Analysis and Applications 24: 182-190

Chen J., Zhao R.L., Li Z.L. (2000) Describing spatial relations with a Voronoi diagram-based nine intersection model. In: Proceedings of 9th international symposium on spatial data handling (SDH 2000). Beijing, China

Chen J., Li Z., Gold C. (2001) A Voronoi-based 9-intersection model for spatial relations. International Journal of Geographical Information Science 15(13): 201- 220

Cheng T., Molenaar M., Bouloucos T. (1997) Identification of fuzzy objects from field objects. In: Spatial information theory, A theoretical Basis for GIS, Lecture notes in computer sciences *1327,* COSIT'97, Springer-Verlag, Germany pp 241-259

Cheng T., Molenaar M., Lin H. (2001) Formalizing fuzzy objects from uncertain classification results. International Journal of Geographical Information Science 15(1): 27- 42

Clementini E., Di Felice P. (1996) An algebraic model for spatial objects with indeterminate boundaries. In: Burrough P., Frank A. (eds) Geographic objects with indeterminate boundaries, Taylor & Francis pp 155-169.

Clementini E., Di Felice P. (1997) Approximate topological relations. International Journal of Approximate Reasoning. Http://www.ing.univaq.it/hpage/eliseo/approx.html

Cohn A.G., Gotts N.M (1996) The `Egg-Yolk' representation of regions with indeterminate boundaries. In: Burrough P., Frank A. (eds) Geographic objects with indeterminate boundaries, Taylor & Francis pp.171-187

Cohn A.G., Bennett B., Gooday J., Gotts N.M. (1997) Representation and reasoning with qualitative spatial relations about regions. In: Stock O. (eds) Spatial and temporal reasoning, Kluwer Publishing Company pp 97-134

Dutta S. (1991) Topological constraints: A representational framework for approximate spatial and temporal reasoning. In: Gunther, Schek (eds) Advances in spatial databases, second symposium, SSD'91, Lecture notes in computer science 525, Springer-Verlag, New yolk pp 161-180

Egenhofer M.J. (1989) A formal definition of binary topological relationships. In: Litwin W., Schek H.J. (eds.) The third international conference of foundations of data organization and algorithms (FODO), Paris, France, Springer-Verlag pp 452-472

Egenhofer M.J. (1994) Deriving the composition of binary topological relations. Journal of Visual Languages and Computing 5: 133-149

Egenhofer M. J., Herring, J.R. (1990) A mathematical framework for the definition of topological relationships. In: Brassel K., Kishimoto H. (eds) Proceedings of the 4th international symposium on spatial data handling, Zurich, Columbus, OH, International Geographical Union pp 803-813.

Egenhofer M. J., Franzosa R. (1991) Point-set topological spatial relations. International Journal of Geographic Information Systems 5(2): 161-174

Egenhofer M.J., Sharma J. (1993) Assessing the consistency of complete and incomplete topological information. Geographic Systems 1: 47-68

Egenhofer M. J., Franzosa R. (1994) On the equivalence of topological relations. International Journal of Geographical Information Systems 8(6): 133-152

Egenhofer M. J., Clementini E., Di Felice P. (1994a) Topological relations between regions with holes. International Journal of Geographic Information Systems 8(2):129-142

Egenhofer M. J., Clementini E., Di Felice P. (1994b) Evaluating inconsistencies among multiple representations. In: Waugh T.C., Healey, R.G. (eds) Advances in GIS research, Proceedings of the 6th international symposium on spatial data handling, Vol. 2, IGU Commission on GIS and Association for Geographic Information pp 901-920

Fisher P. (1996) Boolean and fuzzy region. In: Burrough P., Frank, A. (eds) Geographic objects with indeterminate boundaries, Taylor & Francis pp.87-94

Herring J. (1991) The mathematical modeling of spatial and non-spatial information in geographic information systems. In: Mark D., Frank, A. (eds) Cognitive and linguistic aspects of geographic space, Kluwer Academic Publishers pp 313-350

Ibanez C.E., Tarres J. (1997) On the boundary of fuzzy sets. Fuzzy sets and Systems 89: 113-119.

Kainz W. (1995) Logical Consistency. In: Guptill S., Morrison J. (eds.) Elements of spatial data quality, Elsevier

Kainz W., Egenhofer M.J., Greasley I. (1993) Modeling spatial relations and operations with partially ordered sets. International Journal of Geographic Information systems 7(3): 215-229

Liu Y.M., Luo M.K. (1997) Fuzzy Topology. World Scientific

Molenaar M. (1996) A syntactic approach for handling the semantics of fuzzy spatial objects. In: Burrough P., Frank A. (eds) Geographic objects with indeterminate boundaries, Taylor & Francis pp 207-224

Molenaar M. (1998) An Introduction to the Theory of Spatial Object Modeling, Taylor & Francis.

Van Oosterom P. (1997) Maintaining consistent topology including historical data in a large spatial database. Auto-Carto 13: 327-336, Seattle, WA

Pu P.M., Liu Y.M. (1980) Fuzzy topology I -- neighborhood structure of a fuzzy point and Moor-Smith convergence. Journal of Mathematical Analysis and Applications 76: 581-599

Randell D.A, Cui Z., Cohn A.G. (1992) A spatial logic based on regions and connection. In: Kaufmann M., San Mateo (eds) Proceedings 3rd international conference on knowledge representation and reasoning pp 165-176

Schneider M. (1999) Uncertainty management for spatial data in databases: fuzzy spatial data types. In: The 6th international symposium on advances in spatial databases (SSD), LNCS 1651, Springer Verlag pp 330-351

Tang X.M., Kainz W. (2001) Topological relations between fuzzy regions in a fuzzy topological space. Submitted

Warren H.R. (1977) Boundary of a fuzzy set. Indiana University Mathematics Journal 26(2): 191-197

Wong C.K. (1974) Fuzzy points and local properties of fuzzy topology. Journal of Mathematical Analysis and Applications 46: 316-328

White M.S (1980) A survey of the mathematics of maps. In: Proceedings of Auto Carto IV 1: 82-96

Winter S. (2000) Uncertain topological relations between imprecise regions. International Journal of Geographical Information Science 14(5): 411-430

Wu G. Q., Zheng C.Y. (1991) Fuzzy boundary and characteristic properties of order-homomorphisms. Fuzzy Sets and Systems 39: 329-337

Zhan F.B. (1997) Topological relations between fuzzy regions. In: Bryant B., Carroll J., Oppenheim D., Hightower J., George K. M. (eds.) Proceedings of the 1997 ACM symposium on applied computing, San Jose, CA pp 192-196

Appendix: 44 relations between two simple regions

Matrix	Illustration	Matrix	Illustration	Matrix	Illustration
(1) $\begin{Bmatrix}0,0,1\\0,0,1\\1,1,1\end{Bmatrix}$		(2) $\begin{Bmatrix}0,0,1\\0,1,1\\1,1,1\end{Bmatrix}$		(3) $\begin{Bmatrix}0,1,1\\0,1,1\\1,1,1\end{Bmatrix}$	
(4) $\begin{Bmatrix}0,1,0\\0,1,1\\1,1,1\end{Bmatrix}$		(5) $\begin{Bmatrix}0,1,0\\0,1,0\\1,1,1\end{Bmatrix}$		(6) $\begin{Bmatrix}0,0,1\\1,1,1\\1,1,1\end{Bmatrix}$	
(7) $\begin{Bmatrix}0,0,1\\1,1,1\\0,1,1\end{Bmatrix}$		(8) $\begin{Bmatrix}0,0,1\\1,1,1\\0,0,1\end{Bmatrix}$		(9) $\begin{Bmatrix}0,1,1\\1,1,1\\1,1,1\end{Bmatrix}$	
(10) $\begin{Bmatrix}0,1,0\\1,1,1\\1,1,1\end{Bmatrix}$		(11) $\begin{Bmatrix}0,1,0\\1,1,0\\1,1,1\end{Bmatrix}$		(12) $\begin{Bmatrix}0,1,1\\1,1,1\\0,1,1\end{Bmatrix}$	
(13) $\begin{Bmatrix}0,1,1\\1,1,1\\0,0,1\end{Bmatrix}$		(14) $\begin{Bmatrix}0,1,0\\1,1,1\\0,1,1\end{Bmatrix}$		(15) $\begin{Bmatrix}0,1,0\\1,1,1\\0,0,1\end{Bmatrix}$	
(16) $\begin{Bmatrix}0,1,0\\1,1,0\\0,1,1\end{Bmatrix}$		(17) $\begin{Bmatrix}0,1,0\\1,1,0\\0,0,1\end{Bmatrix}$		(18) $\begin{Bmatrix}1,1,1\\1,1,1\\1,1,1\end{Bmatrix}$	
(19) $\begin{Bmatrix}1,1,0\\1,1,1\\1,1,1\end{Bmatrix}$		(20) $\begin{Bmatrix}1,1,0\\1,1,0\\1,1,1\end{Bmatrix}$		(21) $\begin{Bmatrix}1,1,1\\1,1,1\\0,1,1\end{Bmatrix}$	
(22) $\begin{Bmatrix}1,1,1\\1,1,1\\0,0,1\end{Bmatrix}$		(23) $\begin{Bmatrix}1,1,0\\1,1,1\\0,1,1\end{Bmatrix}$		(24) $\begin{Bmatrix}1,1,0\\1,1,0\\0,1,1\end{Bmatrix}$	
(25) $\begin{Bmatrix}1,1,0\\1,1,1\\0,0,1\end{Bmatrix}$		(26) $\begin{Bmatrix}1,1,0\\1,1,0\\0,0,1\end{Bmatrix}$		(27) $\begin{Bmatrix}1,0,0\\1,1,0\\1,1,1\end{Bmatrix}$	
(28) $\begin{Bmatrix}1,0,0\\1,1,1\\1,1,1\end{Bmatrix}$		(29) $\begin{Bmatrix}1,0,0\\1,1,1\\0,1,1\end{Bmatrix}$		(30) $\begin{Bmatrix}1,0,0\\1,1,0\\0,1,1\end{Bmatrix}$	
(31) $\begin{Bmatrix}1,0,0\\1,1,1\\0,0,1\end{Bmatrix}$		(32) $\begin{Bmatrix}1,0,0\\1,1,0\\0,0,1\end{Bmatrix}$		(33) $\begin{Bmatrix}1,1,1\\0,1,1\\0,0,1\end{Bmatrix}$	
(34) $\begin{Bmatrix}1,1,1\\0,1,1\\0,1,1\end{Bmatrix}$		(35) $\begin{Bmatrix}1,1,0\\0,1,1\\0,1,1\end{Bmatrix}$		(36) $\begin{Bmatrix}1,1,0\\0,1,1\\0,0,1\end{Bmatrix}$	
(37) $\begin{Bmatrix}1,1,0\\0,1,0\\0,1,1\end{Bmatrix}$		(38) $\begin{Bmatrix}1,1,0\\0,1,0\\0,0,1\end{Bmatrix}$		(39) $\begin{Bmatrix}1,0,0\\1,0,0\\1,1,1\end{Bmatrix}$	
(40) $\begin{Bmatrix}1,1,1\\0,0,1\\0,0,1\end{Bmatrix}$		(41) $\begin{Bmatrix}1,0,0\\0,1,0\\0,0,1\end{Bmatrix}$		(42) $\begin{Bmatrix}1,0,0\\0,1,0\\0,1,1\end{Bmatrix}$	
(43) $\begin{Bmatrix}1,0,0\\0,1,1\\0,0,1\end{Bmatrix}$		(44) $\begin{Bmatrix}1,0,0\\0,1,1\\0,1,1\end{Bmatrix}$