The Dimensional Model: A Framework to Distinguish Spatial Relationships

Roland Billen[1], Siyka Zlatanova[2], Pierre Mathonet[3], and Fabien Boniver[4]

Geography Department, University of Liege, Allée du 6-Août 17, 4000 Liège, Belgium, +32-(0)4/3665752, +32-(0)4/36656 93, Roland.billen@ulg.ac.be[1]
Department of Geodesy, Delft University of Technology, Thijsseweg 11, 2629 JA Delft, The Netherlands, +31 (0)15 278 2714, +31 (0)15 278 2745, S.Zlatanova@citg.tudelft.nl[2]
Math. Dept., University of Liege, Grande Traverse 12, 4000 Liège, Belgium, +32-(0)4/3669480, +32-(0)4/366 9547, P.Mathonet@ulg.ac.be[3]
Math. Dept., University of Liege, Grande Traverse 12, 4000 Liège, Belgium, +32-(0)4/3669417, +32-(0)4/366 9547, F.Boniver@ulg.ac.be[4]

Abstract

A unique characteristic of GIS as compared to other information systems, is their capacity to manage *spatial* relationships, such as connections or interrelations among objects in the geometric domain. A number of frameworks use topology as a basic mechanism to define spatial relationships. The OpenGIS consortium has adopted one of them, i.e. the 9-intersection model. In this paper, a new framework for representing spatial relationships - the *Dimensional model* - is introduced. The model was first developed for convex spatial objects and is now extended to topological n-manifolds. It is based on two major concepts, i.e. the *dimensional elements* of spatial objects and the *dimensional relationships, i.e.* the relationships existing between dimensional elements. The model addresses a substantial group of spatial relationships and provides a flexible framework to consider either generalised or specialised types of associations..
Keywords: spatial relationships, spatial model, convexity, topological manifolds

1 Introduction

The development of a mathematical theory for categorising relationships has been identified as an essential task in tackling the diversity and incompleteness of spatial-relationships' representations (see Boyle *et al.* 1983, NCGIA 1989). Among all the approaches (e.g. metrics, topology, ordered sets), topology seems

the most appreciated (see Egenhofer *et al.* 1994, Egenhofer 1989, Egenhoher and Sharif 1998, Clementini *et al.* 1993, Kainz *et al.* 1993, Molenaar 1998). The OpenGIS consortium has adopted one of the topologically based frameworks, referred to as the 9-intersection model, as a generic mechanism for implementation and development.

In this paper, a new framework for representing spatial relationships named the *Dimensional Model* (DM) is introduced. The model was first developed for convex spatial objects and is now extended to topological n-manifolds. The paper is organised in the following sections. First, the order concept for convex bodies is presented. Then, the order concept for a topological manifold is developed. A formal description of the Dimensional Model follows. After a brief comparison with the 9-intersection model, some spatial relationships are depicted using the DM. Finally; the implementation of the DM is overviewed.

2 An Order Formula for Convex Bodies

2.1 Affine Subspaces and Convex Sets

Geographers and mathematicians use co-ordinates to describe location in space. Whenever d is a positive integer, we denote by $\mathbf{R}^d$ the set of tuples $(\alpha_1,...,\alpha_d)$ of real numbers $\alpha_1,...,\alpha_d$. As Euclidean vector (or affine) space, $\mathbf{R}^d$ is the natural framework in which geometry can formally be studied. It is also the first example of Euclidean topological space. A detailed introduction to general topology and manifolds can be found in Alexandrov (1998) and Lee (2000).

A subset A of $\mathbf{R}^d$ is an *affine subspace* if, for any distinct points x, y belonging to A, the (infinite) straight line defined by x and y lies in A. Points, straight lines, planes, and $\mathbf{R}^3$ itself are the only affine subspace of $\mathbf{R}^3$. Their respective *dimensions* are 0, 1, 2 and 3. An affine subspace of dimension d-1 of $\mathbf{R}^d$ is named *hyperplane*. For instance, the hyperplanes are merely straight lines in $\mathbf{R}^2$ and planes in $\mathbf{R}^3$. A hyperplane divides the whole space in two regions, called *halfspaces*.

A subset A of $\mathbf{R}^d$ is *convex* if, for any two points x,y belonging to A, the segment $[x,y]$ lies in A.

A *supporting hyperplane M* of a convex set C is a hyperplane such that

- C is included in one of the halfspaces defined by M,
- $M \cap C \neq \varnothing$.

Fig. 1 shows examples of convex sets and supporting hyperplanes in $\mathbf{R}^2$ and $\mathbf{R}^3$.
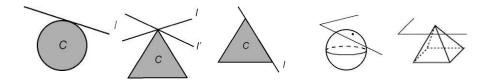
Fig. 1. Examples of closed convex sets and hyperplanes

2.2 Order of Points in Closed Convex Set

Let C be a convex set, closed with respect to the Euclidean topology. Each point of C has an *order*, whose definition is given in Berger (1978, p. 50), and can be stated as follow.

Let C be a closed convex set in $\mathbf{R}^d$ and $x \in C$. The order of x in C, denoted by $o(x,C)$, is the dimension of the intersection of all supporting hyperplanes containing x.

In particular, if no supporting hyperplanes contains x, then x has order d. One can prove that those points with order d are exactly the interior points of C with respect to the Euclidean topology.

Fig. 2 shows points with various orders in a triangle, a drop, and a segment. For instance, the top point of the triangle has order 0. Indeed, infinitely many supporting hyperplanes (only two are represented) go through it and intersect in that point itself. Fig. 3 shows examples in $\mathbf{R}^3$.
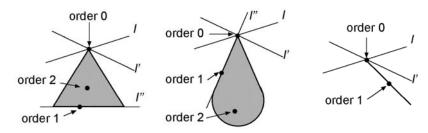

Fig. 2. Order of points of closed convex sets in $\mathbf{R}^2$ (with some hyperplanes)
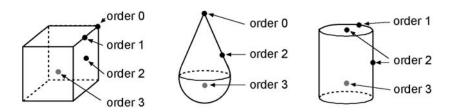

Fig. 3. Order of points of closed convex sets in $\mathbf{R}^3$ (without hyperplanes)

3 An Order Formula for Manifolds

More often than not, objects in GIS application environments are not convex. It is therefore reasonable to seek for a suitable formula for more general objects. Those will be *topological manifolds*. Our approach goes as follow: given a point *x* in a topological manifold, we create a closed convex neighbourhood of *x* and compute its order with respect to this neighbourhood. The key point is that, intuitively, the order of a point is a "local" property: it actually depends only on the manifold in a neighbourhood of *x*

Therefore, the steps to find the order of points are: 1) determine if the point of interest stands on the interior or the boundary of the manifold; 2) centre a ball on this point with a given radius *r*; 3) take the intersection between the ball and either the manifold or the boundary of the manifold; 4) determine the convex hull of this intersection; 5) determine the order of the point regarding the convex object created (the convex hull); 6) repeat the operation 2 to 6 with a smaller radius, until getting a minimum value for the order.

3.1 Topological Manifolds

Let *n* be a positive integer. We denote by $\mathbf{R}^n_+$ the subset of tuples $(\alpha_1,...,\alpha_n)$ with $\alpha_n \geq 0$. A subset *A* of $\mathbf{R}^d$ is a *topological n-manifold with boundary* if each point $x \in A$ has neighbourhood, which is homeomorphic to an open subset of $\mathbf{R}^n_+$.

Let *A* be such *n*-manifold. It is the disjoint union of its *interior*, $A°$, and its *boundary*, ∂A . The points belonging to $A°$ are named *interior points*. A point is interior if it has a neighbourhood homeomorphic to an open subset of $\mathbf{R}^n$. It is worth noticing that those definitions do not coincide with the Euclidean topological definitions of interior and boundary.

Two additional notations will be useful. If *B* is a subset of $\mathbf{R}^d$, we denote by *conv(B)* the *convex hull* of *B*, which is the smallest convex subset of $\mathbf{R}^d$ containing *B*. If *r* is a positive real number and $x \in \mathbf{R}^d$, we denote by $b_{x,r}$ the closed ball with centre *x* and radius *r*.

3.2 The General Formula

We can now write the general formula that encapsulates the algorithm presented above. Let *X* be a topological manifold. Assume furthermore that *X* is closed as a subset of $\mathbf{R}^d$. If *x* is an interior point of *X*, then its *order* is defined as:

$$\lim_{r \to 0, r \neq 0} o(x, conv(b_{x,r} \cap X)) \text{. (1)}$$

Now, if $x \in \partial X$ (*x* is a boundary point of *X*), then its *order* is:

$$\lim_{r \to 0, r \neq 0} o(x, conv(b_{x,r} \cap \partial X)) \text{. (2)}$$

In both cases, the limit exists and equals the minimum of the values of the function *o*.

Fig. 4 illustrates this approach for 1-manifold. Figs. 5 and 6 show some more examples for 1-manifold (curve line) and 2-manifold.
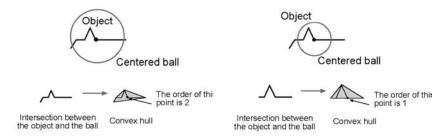

Fig. 4. Determination of the order of a broken line's point with different ball radius
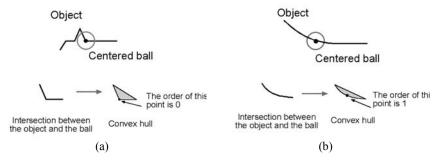

Fig. 5(a) Determination of the minimum order of a broken line's point, **(b)** determination of the minimum order of a curve line's point
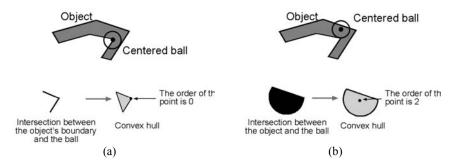

Fig. 6(a) Determination of the minimum order of a polygon's boundary point, **(b)** determination of the minimum order of a polygon's interior point

4 The Dimensional Model

The Dimensional Model (DM) is a (conceptual) framework to describe both spatial objects and spatial relationships. The spatial objects are composed by *dimensional elements,* which are based on order's points of object. The spatial relationships between spatial objects are described in terms of *dimensional relationships*, i.e. relationships that exist between the dimensional elements of the objects.

4.1 Spatial objects in DM

In our model, *a simple spatial object of dimension d is equivalent to a topological d-manifold*. They are called simple because it is possible to apply directly the order formula to them, and therefore determine their dimensional elements. We also define a complex spatial object (as a combination of simple spatial object), which will not be discussed here (see Fig. 7).
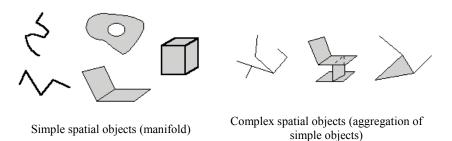

Simple spatial objects (manifold)

Complex spatial objects (aggregation of simple objects)

Fig. 7. Examples of spatial objects

4.2 Dimensional elements of DM

The *dimensional elements* are associated with different parts (or points) of a spatial object according to their order.

1. The α–dimensional element (denoted αD-element) of a spatial object C (which has at least dimension α), corresponds to the set of all the points (or parts) of C which have order 0 to α.
2. The αD-element of a spatial object C has an **extension** and may have a **limit**.
3. The extension is the subset of C formed by its points of order α, and the limit is the subset of C formed by its points of order 0 to order (α-1).

Thus, if the αD-element has a limit, this limit corresponds to a lower (α-1)D-element. The 0D-element does not have a limit by definition. Fig. 8 illustrates the dimensional elements of a polygon. First, the order of all the points is determined.

This convex is composed of 2D, 1D and 0D-elements. The different extension and limits are presented in the Fig. 8. In the case of an ellipse, the 1D-element does not have a limit. It should be noted that there is one and only one *x*D-element for this object.
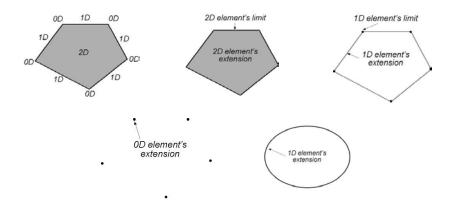

Fig. 8. Order of points and dimensional elements of a polygon and of an ellipse

4.3 Dimensional Relationships

The *dimensional relationships* are defined as the relationships existing between dimensional elements. These relationships can either be total, partial or non-existent, and are oriented (from one element to an other one).

1. A dimensional element is in **total relation** with another dimensional element if their intersection is equal to the first element, and if the intersection between their extensions is not empty.
2. A dimensional element is in **partial relation** with another dimensional element if their intersection is not equal to the first element, and if the intersection between their extensions is not empty.
3. A dimensional element is in **no relation (non-existent)** with another dimensional element if the intersection between their extensions is empty.

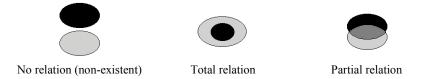Fig. 9 illustrates the three types of dimensional relationships for 2D-elements.

No relation (non-existent)   Total relation   Partial relation

Fig. 9. The different types of dimensional relationships between two 2D-elements (from black element to grey element)

4.4 The Dimensional Model for Investigation of Spatial Relationships

The dimensional elements (with their limits and extensions) and the dimensional relationships (i.e. total, partial and non-existent) are the basic tools to decode spatial relationships. The spatial relationship between two objects can be expressed by the dimensional relationships that exist between the dimensional elements of both objects. For example, let us consider a polygon A (with 2D, 1D, 0D elements) and a line B (with 1D and 0D-elements). The dimensional relationships between the spatial object A and the spatial object B can be identified in the following sequence: first, check the dimensional relationship between the 2D-element of A and all the dimensional elements of spatial object B; then, check the dimensional relationship between the 1D-element of A and all the dimensional elements of spatial object B, etc. The dimensional relationships between B and A can be found following the same approach. Three groups of dimensional relationships can be distinguished following this approach, i.e. the *simplified*, the *basic* and the *extended* relationships.

A dimensional relationship is coded *RnDy*, using the notations R for relationships, *nD* for dimension of the element of the first object, and *y* dimension of the element of the second object. R2D1 represents the dimensional relationships between the 2D-element of the first object and the 1D-element of the second object. Furthermore, a numeric code for the three types of dimensional relationships, i.e. 0 for non-existent, 1 for total and 2 for partial, is specified.

The basic relationships. This group contains all the relationships between every possible combination of dimensional elements. For example, the spatial relationship between a polygon (considering 2D, 1D, 0D elements) and a line (considering 1D and 0D elements) can be expressed by basic dimensional relationships as follows:

R2D1	R2D0	R1D1	R1D0	R0D1	R0D0
{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}	{0,1,2}

where 0, 1, 2 correspond to the possible dimensional relationships, i.e. non-existent, total and partial.

The extended relationships. The partial relation can be further investigated for the dimension of the intersection. For example, in $\mathbf{R}^3$, a 2D-element and a 1D-element may have a 1D- or a 0D-intersection. If the intersection has the same dimension as the lowest dimensional element in the relation, it keeps the code 2 (e.g., if a 2D-element and a 1D-element have a 1D-intersection, it would be noted as R2D1 2). If the dimension of the intersection is just inferior, then it would have code 3 (a 0D-intersection in the example, R2D1 3). Our example of the extended dimensional relationships between a polygon and a line becomes:

R2D1	R2D0	R1D1	R1D0	R0D1	R0D0
{0,1,2,3}		{0,1,2}	{0,1,2,3}	{0,1,2}	{0,1,2}	{0,1,2}

The simplified dimensional relationships. In many cases the dimensional elements of the second object is not relevant. For example, it might be interesting

to know if the 2D-element of a polygon has a relationship with another object independently of its dimensional elements. In such cases, some dimensional relationships can be aggregated. The complete aggregation rules will not be exposed here. Our example of the simplified dimensional relationship becomes:

```
R2D    R1D      R0D
{0,1,2} {0,1,2}  {0,1,2}
```

Fig. 10 illustrates how the spatial relationships between a polygon and a line are represented according to the different groups in the Dimensional Model.
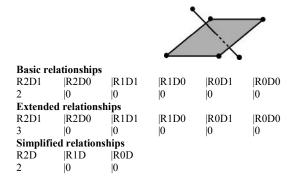

Basic relationships

R2D1	|R2D0	|R1D1	|R1D0	|R0D1	|R0D0
2	|0	|0	|0	|0	|0

Extended relationships

R2D1	|R2D0	|R1D1	|R1D0	|R0D1	|R0D0
3	|0	|0	|0	|0	|0

Simplified relationships

R2D	|R1D	|R0D
2	|0	|0

Fig. 10. The Dimensional Model applied to the relationship between polygon and line

5 The Dimensional Model Versus the 9-intersection Model

As mentioned, the 9-intersection model is a one of the topological relationships standard. We want to clearly expose the difference that exists between the 9-intersection model and the Dimensional Model.

5.1 The 9-intersection Model (9i Model)

The framework is based on the assumptions of spatial objects (represented by 0,1,2,3-cells) without holes and intersecting parts. If the spatial objects A and B are defined in the same topological space, their *boundary, interior* and *exterior* are denoted by $\partial A, A^\circ, A_-, \partial B, B^\circ$ and B^- (see Egenhofer, M. and J. Herring, 1990). The binary relationship $R(A,B)$ between the two objects is then identified by composing all the possible set intersections of the six topological primitives, i.e. $\partial A \cap \partial B$, $A^\circ \cap B^\circ$, $\partial A \cap B^\circ$, $A^\circ \cap \partial B$, $A^- \cap B^-$, $A^- \cap \partial B$, $A^- \cap B^\circ$, $\partial A \cap B^-$ and $A^\circ \cap B^-$, and detecting empty (0) or non-empty (1) intersections. For example, if two objects have a common boundary, the

intersection between the boundaries is non-empty, i.e. $\partial A \cap \partial B = 1$; if they have intersecting interiors, then the intersection $A° \cap B°$ is not empty, i.e. $A° \cap B° = 1$. To represent the relationships a decimal coding is adopted here (see Kufoniyi 1995). That is to say, that the binary number (obtained from a particular ordering of all the intersections) is converted into a decimal number. For example, the relationship that has a binary number 000011111 (considering the ordering given above) equals decimal number 31 and hence the relationship is R031 (disjoint). Thus, all the decimal codes are between R000 and R512

5.2. Differences Between 9i Model and DM

The differences between the 9-intersection model and DM can be summarised as follows:

1. The definition space of 9i model is topological when it is an affine one for DM.
2. In the 9i model, the spatial relationships are determined looking to intersections of the topological primitives. In the DM, the spatial relationships are determined looking to intersections of the dimensional elements.
3. The 9i model is related to the cell approach. The dimension of the cells are not (always) equivalent to order of points (see Fig. 11). Therefore, the union of d-cells of an object is not equivalent to its dD element, except for some types of objects (as polytopes).

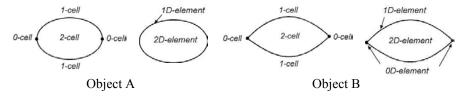Object A   Object B

Fig. 11. Examples of differences between cell decomposition and dimensional elements

4. The DM allows grouping spatial relationships at different level of complexity: more general groups than with 9i model, same kind of groups or more specific ones (see the next section).

6 Possible Spatial Relationships

As mentioned, three groups of dimensional relationships can be used to express the spatial relationship between objects. Furthermore, one has the choice to take into consideration only relevant dimensional elements in a geographical perspective. For example, a particular geographical phenomena represented by a polygon may not need a distinction between the 1D-element and the 0D-element which form its border. In such a case, although the 0D-element exists in the

object's definition, it would not be taken into account in the determination of the spatial relationship.

The number of potential relationships between two objects depends, with respect to the Dimensional Model, on: 1) the dimensional nature of the objects (given by dimensional elements), 2) the semantic dimension of the object (only the "relevant" dimensional elements from a semantic point of view) and 3) the group of dimensional relationships. Similarly to the 9-intersection model, only a small number of the theoretical relationships can be realised in reality (see Zlatanova, 2000 for examples in $\mathbf{R}^3$). The same approach is adopted, i.e. elimination of impossible relationships by negative conditions. All the possible relationships between line-line, line-surface, line-body, surface-surface, surface-body and finally body-body have been established and studied for the different criterion mentioned above. Note that in this study, some elements have been simplified. For example the 0D element of a line corresponds only to its extremities (no broken lines). Table 1 portrays simplified, basic and extended relationships for all levels of dimensional relevancy.

Table 1. Possible relationships according to the Dimensional Model

D element	Dim. Rel.	Line-Line	Surface-line	Body-body	Surface-surface	Body-Line	Body-surface
(n)D	S.	5	3	5	5	3	3
	B.	5	3	5	5	3	3
	E.	7	5	5	11	3	3
(n)D	S.	11	10	8	15	6	8
&(n-1)D	B.	33	31	8	43	19	19
	E.	61	?	15	?	43	48
(n)D	S.		?	?	?	19	?
&(n-1)D	B.		?	?	?	?	?
&(n-2)D	E.		?	?	?	?	?
(n)D	S.			?		?	?
&(n-1)D	B.			?		?	?
&(n-2)D	E.			?		?	?
&(n-3)D							

With Dim. Rel. = Dimensional relationship; S. = simplified; B. = basic; E. =extended

?   non determinate

     impossible case

33   possible relationships according to the 9-intersection model

Considering the highest and the second highest dimensional element and using the basic relationship, the relationships reported by Zlatanova (2000) are found. They are shown in bold font in table 1 (light grey row). Most of the equivalent topological cases can be "refined" using some of the more complex criteria in the Dimensional Model (i.e. everything that is below the shaded line in table 1). Further, more aggregated relationships can be found with simpler criterions (i.e. everything that is above this line). Fig. 12 presents the extended dimensional solutions (0D elements are not taken into account) to the topological equivalence R095 and R287 (between a body *A* and a surface *B*).
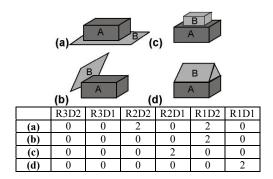
	R3D2	R3D1	R2D2	R2D1	R1D2	R1D1
(a)	0	0	2	0	2	0
(b)	0	0	0	0	2	0
(c)	0	0	0	2	0	0
(d)	0	0	0	0	0	2

Fig. 12. Dimensional relationships for, **(a)** and **(b)** topological equivalence R095, and **(c)** and **(d)** R287

Another interesting example concerns the "meet" relationship between two bodies. All the spatial situations in Fig. 13 are equivalent to R287 according to the 9-intersection model, while all of them can be differentiated with the Dimensional model. Solutions are given only for cases *a, b, c* and *d*. The cases *e* and *f* need to consider the 0D-element.

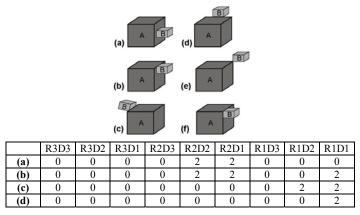|	R3D3	R3D2	R3D1	R2D3	R2D2	R2D1	R1D3	R1D2	R1D1
(a)	0	0	0	0	2	2	0	0	0
(b)	0	0	0	0	2	2	0	0	2
(c)	0	0	0	0	0	0	0	2	2
(d)	0	0	0	0	0	0	0	0	2

Fig. 13. Dimensional relationships for topological equivalence (R287)

7 Implementation of the DM

The DM is oriented spatial relationships. . It is possible, however, to use it to enrich spatial data structures and some convenient tests have been done. The "purest" solution is to use D elements instead of topological primitives. Of course, some problems may occur when objects are transformed – the D elements are not invariant through all the homeomorphic transformations. A more pragmatic

solution is to keep a topological data structure and compute the dimensional elements during the determination process of dimensional relationships. This computation uses colinearity and coplanearity algorithms. The search for an adapted implementation of DM has revealed some very interesting concepts, such as the possibility of relinquishing the *single-valued map* approach (see Molenaar, 1998). This is still being considered and will not be addressed here.

8 Conclusions

In this paper, we have presented a new framework, i.e. the Dimensional Model, to distinguish spatial relationships of spatial objects. The Dimensional Model supports the presentation of a very large group of spatial relationships. It provides a flexible framework that allows either generalised or specialised types of relationships to be considered. The freedom in choosing geographically relevant dimensional elements and groups of dimensional relationships allows one to decide on a particular complexity of spatial relationships. It covers a large range of spatial objects (topological n-manifold for simple spatial objects and combinations of simple spatial objects for complex spatial objects).

Acknowledgements

References

Alexandrov PS (1998) Combinatorial topology, Vol. 1, 2 and 3. Dover Publications Inc., Mineola, New York

Boyle A, Dangermond J, Marble D, Simonett D, Smith L, Tomlinson R (1983) Final report of a conference on the review and synthesis of problems and directions for large scale geographic information system development. Technical report, NASA NAS2-11246

Berger M (1978) Géométrie 3 : convexes et polytopes, polyèdres réguliers, aires et volumes. Paris, Cedic/Fernand Nathan

Clementini E, di Felice P, van Oosterom P (1993) A small set of formal topological relations suitable for end-user interaction. In: Proceedings of the 3rd International Symposium on Large Spatial Databases. Springer-Verlag, Berlin, pp 277-295

Egenhofer M (1989) A formal definition of binary topological relationships. In: Proceedings of the 3rd International Conference on Foundation of Data Organisation and Algorithms. pp 457-472

Egenhofer M, Clementini E, di Felice P (1994) Topological relations between regions with holes. International Journal of Geographical Information Systems 8(2):129-144

Egenhofer M, Herring J (1990) A mathematical framework for the definition of topological relationships. In: Proceedings of Fourth International Symposium on Spatial Data Handling. Zurich, Switzerland, pp 803-813

Egenhofer M, Shariff B (1998) Metric details for natural-language spatial relations. ACM transactions on information systems 16(4):295-321

Kainz W, Egenhofer M, Greasley (1993) Modelling spatial relations and operations with partially ordered sets. International Journal of Geographical Information Systems 7(3):215-229

Kufoniyi O (1995) Spatial coincidence modelling, automated database updating and data consistency in vector GIS. Ph.D. thesis, ITC, The Netherlands

Lee J (2000) Introduction to topological manifolds. New-York, Springer-Verlag

Molenaar M (1998) An introduction to the theory of spatial objects modelling. London, Taylor&Francis

NCGIA (1989) The research plan of the National Center for Geographic Information and Analysis. International Journal of Geographical Information Systems 3(2):117-113

Zlatanova S (2000) On 3D topological relationships. In: Proceedings of the 11th International workshop on Database and Expert System applications. Greenwich, London, pp 913-919