LINE VORONOI DIAGRAM BASED INTERPOLATION AND APPLICATION TO DIGITAL TERRAIN MODELLING

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ABSTRACT

Local coordinates based on the Voronoi diagram are used in natural neighbour interpolation to quantify the "neighbourliness" of data sites. In an earlier paper, we have extended the natural neighbour or stolen area interpolation technique from ordinary Voronoi diagrams to Voronoi diagrams for sets of points and line segments, by providing direct vectorial formulas for the first order and second order derivatives for the stolen area. This generalization allows one to model linear discontinuities, that are not modelable through current interpolation techniques. In this paper, we recall the continuity and differentiation properties of these local coordinates and natural neighbour interpolation, and we present their application to digital terrain modelling. In our case, the data sites can be either points or oriented line segments. We use this natural neighbour interpolation in order to interpolate elevations from the neighbouring data sites, and construct a digital terrain model. We present an example of the use of the natural neighbour interpolation technique based on the Voronoi diagram for a set of points and oriented line segments for digital terrain modelling. The applications of this extended interpolation technique are shown for the modelling of linear vertical faults, dams or bridges. This research brings novelty in the modelling of topographic artifacts represented by line segments (e.g. thalwegs, crests, faults), because in the natural neghbour interpolation technique that we are using, line segments are data objects that can have an elevation (in fact an elevation for each oriented line segment).

1 INTRODUCTION

Local coordinates based on the Voronoi diagram were introduced in (Sibson, 1980), and they have not been extended to generalized Voronoi diagrams until now except in (Anton et al., 1998). Local coordinates based on the Voronoi diagram have been used in natural neighbour interpolation (Sibson, 1981) (also studied in (Gold and Roos, 1994) as stolen area interpolation), to quantify the "neighbourliness" of the data sites. The properties of these local coordinates have been extensively studied in Farin (Farin, 1990) and Piper (Piper, 1993), who gave a formula for the gradient of the volume stolen from neighbouring Voronoi regions due to the insertion of a query point, obtained from two directional derivatives. The natural neighbour or stolen area interpolation technique has been extended from ordinary Voronoi diagrams to Voronoi diagrams for sets of points and line segments in (Anton et al., 1998). Anton et *al.* (Anton et al., 1998) extended the results presented in (Gold and Roos, 1994), by providing direct vectorial formulas for the first order and second order derivatives for the stolen area. The analysis presented in (Anton et al., 1998) generalizes the analysis of Piper (Piper, 1993) based on the formalism of partial derivatives, to the formalism of derivatives of a function on a normed space.

Even though the Voronoi diagram for a set of points and oriented line segments has never been defined formally (i.e. the meaning of oriented line segments), it has been used in Gold (Gold and Roos, 1994). The objects in this generalized Voronoi diagram are either points or pairs of oriented line segments (see section 2.7.2 in (Berger, 1977a) and section 8.6.1 in (Berger, 1977b) for the definition of oriented lines). We review this data structure in Section 2.

The representation of line segments in Digital Terrain Modelling (DTM) has been traditionally done using constrained Delaunay triangulations (Jones and J., 1998, Farestam and Simpson, 1995, Ding, 1995, Li and Wong, 1999). In all these DTMs, the line segments were represented as constrained edges in the Delaunay triangulation (dual of the ordinary Voronoi diagram for a set of points), but their elevation could not be taken into account for the interpolation, because the only objects that had elevations were points (the end points of the line segments). This research brings novelty in the modelling of topographic artifacts represented by line segments (e.g. thalwegs, crests, faults), because in this interpolation technique, line segments are data objects that have an elevation (in fact an elevation for each oriented line segment).

In Section 3, we present the extension of the stolen area interpolation technique from the ordinary Voronoi diagram to the Voronoi diagram for a set of points and line segments. We present direct vectorial formulas for the first order and second order derivatives for the stolen area.

In Section 4, we present the application of this extended natural neighbour interpolation to topographic modelling. In our case, the interpolant is the elevation. In spatial interpolation, local techniques have been used in order to get an interpolation continuous at data points, and smooth around data points. In these local techniques, the data points which influence the interpolant are the ones neighbouring the given interpolation point.

2 THE VORONOI DIAGRAM FOR A SET OF POINTS AND ORIENTED LINE SEGMENTS

Let us first introduce the definition of the Voronoi diagram for a set of sites (i.e. objects or subsets) in the Euclidean affine space of dimension n.

Definition 1. Let $\mathcal{O}$ be a set of sites in Euclidean affine space of dimension n. For each site o of $\mathcal{O}$, the Voronoi cell $V(o)$ of o is the set of points that are closer to o than to other sites of $\mathcal{O}$. The Voronoi diagram $V(\mathcal{O})$ is the space partition induced by Voronoi cells.

Then let us introduce the definition of the Delaunay triangulation of a set of sites (or objects) in the Euclidean space of dimension n.

Definition 2. The Delaunay triangulation of $\mathcal{O}$ is the geometric dual of the Voronoi diagram of $\mathcal{O}$: two sites of $\mathcal{O}$ are linked by an edge in the Delaunay triangulation if and only if their cells are incident in the Voronoi diagram of $\mathcal{O}$.

Let us now consider a set of points and line segments $\mathcal{O}$=$\{O_1, ..., O_s\}$ in the Euclidean plane. The distance from a point M to an object O_i is defined as: either the Euclidean distance between the two points if the object is a point, or $d(M, O_i) := \inf_{P \in O_i} d_e(M, P)$ where d_e denotes the Euclidean distance between two points, otherwise. The Voronoi cell $V(O_i)$ of O_i is the set of points that are closer (in the sense of the distance between a point and an object defined just above) to O_i than to other sites $O_j : j \neq i$ of $\mathcal{O}$. An example of Voronoi diagram for a set of points and line segments is shown in Figure 1.

The Voronoi diagram for a set of points and oriented line segments is a generalized Voronoi diagram. Let's now introduce the definition of a generalized Voronoi diagram (see (Okabe et al., 2000)), in order to be able to introduce the definition of the Voronoi diagram for a set of points and oriented line segments as a generalized Voronoi diagram.

Let S be the space in which we place ourselves (typically $\mathbb{R}^2$). We consider a mapping $\delta : S \times \mathcal{O} \to \{0, 1\}$ defined by $(p, O_i) \mapsto \delta(p, O_i)$ such that

$$\delta(p, O_i) = \begin{cases} 1, & \text{if } p \text{ is assigned to } O_i \\ 0, & \text{otherwise} \end{cases}.$$

We call δ as defined above an assignment rule. Under an assignment rule δ, we consider the set of points assigned to O_i, i.e., $V(O_i) = \{p \mid \delta(p, O_i) = 1, p \in S\}$, and the set of points assigned to both O_i and O_j, with $i \neq j$ i.e., $e(O_i, O_j) = \{p \mid \delta(p, O_i) = \delta(p, O_j) = 1, p \in S\}$. We define the set $\mathcal{V}(\mathcal{O}, \delta, S)$=$\{V(O_1), ..., V(O_n)\}$. Now, let $N_\varepsilon(p)$ be the open ball with radius ε centered at point p.

We restrict the assignment rule δ to satisfy the two following conditions: every point in S is assigned to at least one element of $\mathcal{O}$ i.e., $\forall p \in S, \sum_{i=1}^{n} \delta(p, O_i) \geq 1$; and the set $e(O_i, O_j)$ pertains to the boundary of $V(O_i)$, i.e., $\forall \varepsilon > 0, \forall p \in e(O_i, O_j) : N_\varepsilon(p) \cap [V(O_i) \setminus e(O_i, O_j)] \neq \emptyset$ and $N_\varepsilon(p) \cap [S \setminus V(O_i)] \neq \emptyset$. A set $\mathcal{V}(\mathcal{O}, \delta, S)$ such that the assignment rule δ satisfies the two preceeding conditions is a tessellation. Indeed, the first condition implies that the elements in $\mathcal{V}(\mathcal{O}, \delta, S)$ are collectively

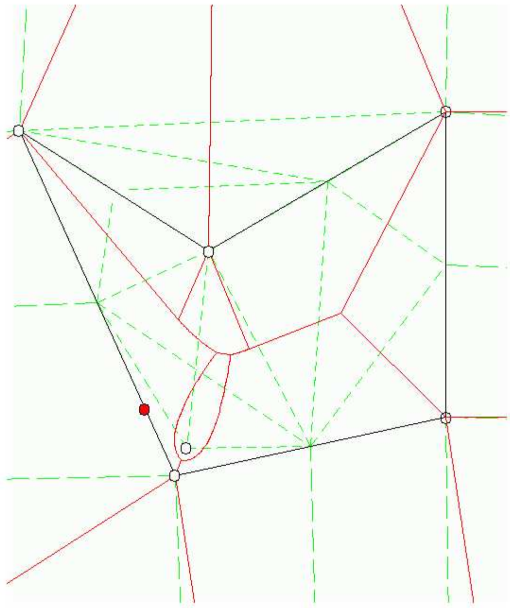

Figure 1: The Voronoi diagram of a set of points and line segments

exhaustive i.e., $\bigcup_{i=1}^{n} V(O_i) = S$.
The definitions of $V(O_i)$ and of $e(O_i, O_j)$ together with the second condition imply that the elements in $\mathcal{V}(\mathcal{O}, \delta, S)$ are mutually exclusive except for boundaries i.e.,
$[V(O_i) \cap V(O_j)] \setminus e(O_i, O_j) = \emptyset$ for all $i \neq j$. We designate this tessellation the generalized Voronoi diagram generated by the generator set $\mathcal{O}$ with assignment rule δ in space S, and $V(O_i)$ the generalized Voronoi region associated with O_i. We call the assignment rule δ that generates a generalized Voronoi diagram, the Voronoi generation assignment rule, or shortly the $V-$assignment rule. The Voronoi diagram for a set of points and oriented line segments in the Euclidean plane is a generalized Voronoi diagram where the space is the Euclidean plane, the generator set is comprised of points and/or pairs of oriented line segments in the Euclidean plane, and the generator assignement rule is as follows.
If O_i is a point, then

$$\delta(p, O_i) = \begin{cases} 1, & \text{if } d(p, O_i) \leq d(p, O_j), \forall j \\ 0, & \text{otherwise} \end{cases}.$$

If O_i is an oriented line segment, then

$$\delta(p, O_i) = \begin{cases} 1, & \text{if } d(p, O_i) \leq d(p, O_j), \forall j \text{ and} \\ & p \text{ is on the left of or on } O_i \\ 0, & \text{otherwise} \end{cases}.$$

3 THE NATURAL NEIGHBOUR INTERPOLATION

In this section, we will make a brief introduction to the natural neighbour interpolation work developed by Anton et al. (Anton et al., 1998). We have a set $O = \{O_1, ..., O_s\}$ of neighbouring data objects, at which we know the elevation, and we want to interpolate the elevation at some unknown location M in the convex hull of O. If the object is a line segment, we know that the elevation varies linearly on each oriented line segment. In order to interpolate the elevation at M from the values at neighbouring data sites, we compute the local coordinates of M.

These local coordinates are defined as follows: $u_k(M) = \frac{\lambda_k(M)}{\sum_i \lambda_i(M)}$, where $\lambda_k(M)$ is the area of the intersection (region marked as V_k on Figure 2) of the “old” tile of O_k and the “new” tile of M.

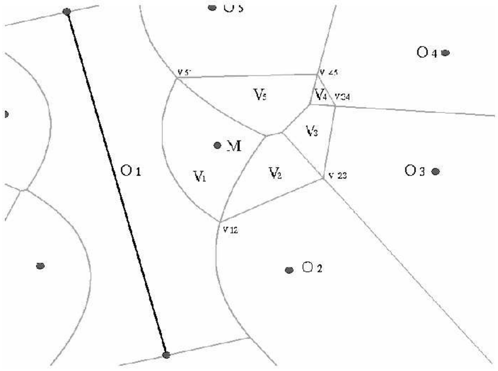

Figure 2: Natural neighbour interpolation

The vectorial expression for the Voronoi vertex (circumcentre) of O_i, O_j, and M is:

$$\overrightarrow{v_{i,i+1}} = \overrightarrow{m_{i,i+1}} + \frac{\overrightarrow{O_{i+1}M} \cdot \overrightarrow{O_iM}}{2\overrightarrow{n_{i,i+1}} \cdot \overrightarrow{O_iM}} \overrightarrow{n_{i,i+1}},$$

where $m_{i,i+1}$ is the middle point of $[O_iO_{i+1}]$, and $n_{i,i+1}$ is defined as follows: $n_{i,i+1} = \begin{pmatrix} O_{i+1,2} - O_{i,2} \\ O_{i,1} - O_{i+1,1} \end{pmatrix}$ where $O_{-,1}$ designates the x coordinate of the object O_- and $O_{-,2}$ designates the y coordinate of the object O_-.

From this expression, we get that the Voronoi vertex is defined, continuous and differentiable except at data sites, and its derivative at the point M is:

$$D\overrightarrow{v_{i,i+1}}(M) = \frac{d\overrightarrow{M} \cdot \overrightarrow{v_{i,i+1}M}}{\overrightarrow{n_{i,i+1}} \cdot \overrightarrow{O_iM}} \overrightarrow{n_{i,i+1}}.$$

Even though, the Voronoi vertex is not continuous at data points, its continuity can be extended at data points since the limit of the Voronoi vertex when the interpolation point goes towards a data point is the point at infinity in the direction of the bisector.

In order to determine $\lambda_k(M)$, we decompose the corresponding area in triangles (see Figure 3): $v_{k-1,k}$, $v_{k,k+1}$, $C_{k,1}$ and $v_{k-1,k}$, $C_{k,j}$, $C_{k,j+1}$, where $C_{k,j}$ is the i^{th} Voronoi vertex of $V(O_k)$ in the counterclockwise orientation from $\overrightarrow{v_{k-1,k}v_{k,k+1}}$ and we get the following result:

$$2\lambda_k(M) = det\left(\overrightarrow{v_{k-1,k}v_{k,k+1}}, \overrightarrow{v_{k-1,k}C_{k,1}}\right)$$

$$+ \sum_{j=1}^{J_k} det\left(\overrightarrow{v_{k-1,k}C_{k,j}}, \overrightarrow{v_{k-1,k}C_{k,j+1}}\right).$$

Therefore, the local coordinates are defined, continuous, and differentiable everywhere except at data sites, and we get:

$$2D\lambda_k(M) = det\left(d\overrightarrow{v_{k-1,k}}, \overrightarrow{C_{k,J_k}v_{k,k+1}}\right) +$$

$$det\left(d\overrightarrow{v_{k,k+1}}, \overrightarrow{v_{k-1,k}C_{k,1}}\right).$$

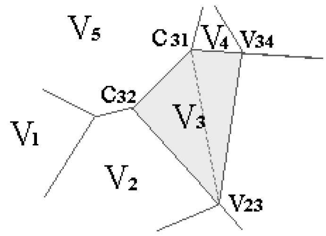

Figure 3: Decomposition into triangles

Even though, the local coordinates are not continuous at data points, their continuity can be extended at data points since the limit of the local coordinate when the interpolation point goes towards a data point is 1.

By the chain rule (see (Cartan, 1967)) $D\lambda_k(M) = \nabla\lambda_k(M) \cdot d\overrightarrow{M}$, we get the direct formula for the gradient of the area stolen to O_k by M.

We extend the local coordinates to line segments, by considering that in the computation of the area stolen by M, we integrate the areas stolen by M, to each portion of length $d\lambda$ of the line segment.

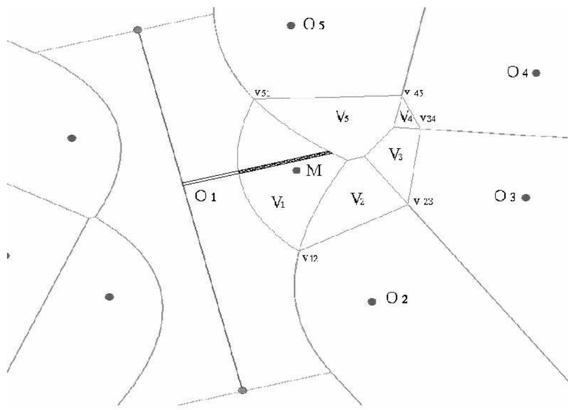

Figure 4: The area stolen to an oriented line segment

The circumcircle corresponding to M, P_i and P_{i+1} (two points on the line O_k separated by $d\lambda$) is: $\overrightarrow{w_{i,i+1}} = \overrightarrow{P_i} + \frac{\overrightarrow{P_iM} \cdot \overrightarrow{P_iM}}{2} \overrightarrow{n} \cdot \overrightarrow{P_iM}\overrightarrow{n}$ where $\overrightarrow{n}$ is a unit vector orthogonal to O_k, and directed from O_k towards M. Its derivative at M is: $D\overrightarrow{w_{i,i+1}}(M) = \frac{d\overrightarrow{M} \cdot \overrightarrow{w_{i,i+1}M}}{\overrightarrow{n} \cdot \overrightarrow{P_iM}} \overrightarrow{n}$. The formula for the area stolen by M depends on the nature (i.e. point or oriented line segment) of the objects O_{k-1} and O_{k+1}. For neighbouring objects that are points, we get:

$$\lambda_k(M) =$$

$$\frac{1}{2}\overrightarrow{n} \cdot \overrightarrow{P_0O_{k-1}}$$

$$\left[\lambda\left(\overrightarrow{P_0O_{k-1}} \cdot \overrightarrow{P_0O_{k-1}}\right) - \lambda^2\left(\overrightarrow{t} \cdot \overrightarrow{P_0O_{k-1}}\right) + \frac{\lambda^3}{3}\right]_{\lambda_{v_{k-1,k}}}^{\lambda_{C_{k,1}}}$$

$$-\frac{1}{2\overrightarrow{n} \cdot \overrightarrow{P_0M}}\left[\lambda\left(\overrightarrow{P_0M} \cdot \overrightarrow{P_0M}\right) - \lambda^2\left(\overrightarrow{t} \cdot \overrightarrow{P_0M}\right) + \frac{\lambda^3}{3}\right]_{\lambda_{v_{k-1,k}}}^{\lambda_{C_{k,1}}} +$$

$$\frac{1}{2\overrightarrow{n} \cdot \overrightarrow{P_0O_{k+1}}}\left[\lambda\left(\overrightarrow{P_0O_{k+1}} \cdot \overrightarrow{P_0O_{k+1}}\right) - \lambda^2\left(\overrightarrow{t} \cdot \overrightarrow{P_0O_{k+1}}\right) + \frac{\lambda^3}{3}\right]_{\lambda_{C_{k,1}}}^{\lambda_{v_{k,k+1}}}$$

$$-\frac{1}{2\overrightarrow{n} \cdot \overrightarrow{P_0M}}\left[\lambda\left(\overrightarrow{P_0M} \cdot \overrightarrow{P_0M}\right) - \lambda^2\left(\overrightarrow{t} \cdot \overrightarrow{P_0M}\right) + \frac{\lambda^3}{3}\right]_{\lambda_{C_{k,1}}}^{\lambda_{v_{k,k+1}}}.$$

4 THE INTERPOLATION PROCESS

In order to interpolate the elevation at a point, the algorithm locates an edge of the triangle of the dual of the Voronoi diagram (i.e. Delaunay triangulation) for a set of points and oriented line segments in which the given point lies. Then, it determines if the given point is a vertex or it lies on the line segment supporting a vertex (oriented half line segment) of the triangle in which it lies. If this is the case, it means that the given point is one of the data points, or it lies on a data line segment. In the first case, the elevation at that point is the same as the elevation at the data point. In the second case, we assume that the elevation at the point can take both of the elevations linearly interpolated from the elevations at the extremities of the two oriented line segments. If it is not the case, the algorithm computes the list of objects (corresponding to Delaunay triangulation vertices) from which the point would steal some area if it was inserted in the Delaunay triangulation. This is done without inserting the point in the Delaunay triangulation. We are using the Quad-edge data structure (Guibas and Stolfi, 1985) for storing both the Delaunay triangulation and the Voronoi diagram. Starting from the located edge, and visiting the three edges of the enclosing triangle, the algorithm tests whether the given edge is safe with respect to the interpolation point (i.e. the edge would remain after the addition of the interpolation point in the Delaunay triangulation (Guibas and Stolfi, 1985)). If an edge is safe, then it is added to the circular list of safe edges enclosing the interpolated point. If an edge is not safe, the edge having the same origin and immediately after it in the clockwise orientation (the edge pointed by its Oprev operator (Guibas and Stolfi, 1985)), and the edge having the same destination and immediately after it in the counterclockwise orientation (the edge pointed by its Dnext operator (Guibas and Stolfi, 1985)) are successively checked. The safe edges detected by this algorithm are inserted in a circular list in the counterclockwise order, and the "previous" pointer points to the previous edge in the counterclockwise orientation. The origin of the edge pointed by the Rot operator (Guibas and Stolfi, 1985) of each edge of this circular list is a natural neighbour of the point being interpolated.

Once the list of enclosing safe edges is computed, the area stolen by the interpolated point to each associated neighbour and the interpolated elevation are computed and the total area and the sum of interpolated elevations are maintained. Once all the neighbours have been visited, the sum of the interpolated elevations is divided by the total area in order to get the interpolated elevation of the point being interpolated. The construction of the Quad-Edge data structure requires $O\left(n \log n\right)$ worst case time, where n is the number of sampled points. Each interpolation requires $O\left(\log n\right)$ amortized worst case time.

5 EXPERIMENTAL RESULTS

In Figures 5,6,7, and 8, we can see the results of the use of our natural neighbour interpolation based on the Voronoi diagram for a set of points and oriented line segments for simulating linear discontinuities like faults, bridges, brakes of slope, dams and faults in topographic modelling. Figures 5 and 6 show the modelling of a vertical fault as the result of the interpolation based on the Voronoi diagram for a set of points and one pair of oriented line segment. Figures 7 and 8 show the use of a smoothing function on top of the interpolation in order to get a smoother surface. The smoothing function we used is an Hermitian interpolation function (Davis, 1975). It applies the following change on the local coordinates producing the new local coordinates: $u'_k\left(M\right) = 3u_k\left(M\right)^2 - 2u_k\left(M\right)^3$. Figure 7 has a different altimetric scale than Figures 5 and 8 in order to show the effect of the smoothing function on the top of the interpolated surface.

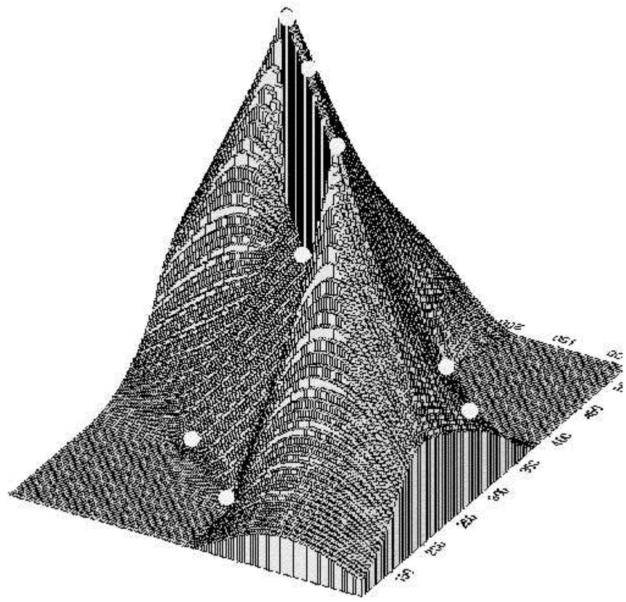

Figure 5: An example of line Voronoi diagram based natural neighbour interpolation, the data objects are marked by plain disks

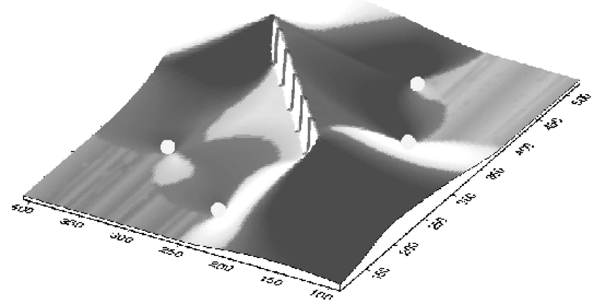

Figure 6: A view of the same surface with illumination (the source is at 30 degrees above in the north east direction)

6 DISCUSSION

We have shown in this paper an extension of the natural neighbour interpolation from the ordinary Voronoi diagram to the Voronoi diagram for a set of points and oriented line segments. We have presented an example of use of this interpolation technique for digital terrain modelling.

7 ACKNOWLEDGMENTS

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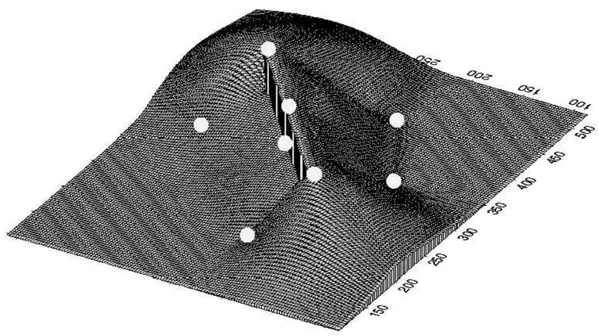

Figure 7: The surface obtained from the same data after using an Hermitian smoothing function, the data objects are marked by plain disks, the scale has been reduced in order to see the whole top surface

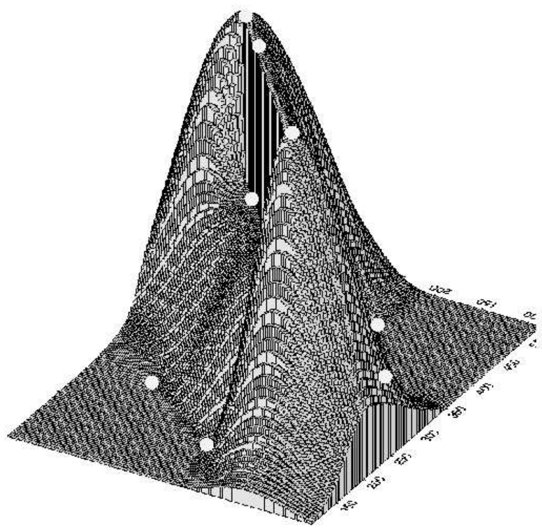

Figure 8: The same surface at a different altimetric scale

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