THE EXPRESSION OF SPHERICAL ENTITIES AND GENERATING OF VORONOI DIAGRAM BASED ON TRUNCATED ICOSAHEDRON DGG

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ABSTRACT:

Based on the analysis of current spherical spatial data structure, the paper discusses the clew of the spherical spatial basal subdivision mode which uses inverse Snyder polyhedron equal area projection on the surfaces of Truncated Icosahedron, and then obtains global multi-resolution overlay by hierarchical subdivision on the initial unfolded projection plane according to the hexagonal grid. The paper also puts forward a essential clew of managing the three leaves nodes and tiles coding based on hexagonal grid, and establish the hexagonal grid expression modes of different spherical entities. Also, on the basis of these clews, we put forward the algorithm of generating the voronoi diagram based on the spherical hexagonal overlay and verified the exactness and efficiency of the algorithm through experiments. By comparing with other similar algorithms, the paper summarized the advantages of the algorithm and gives further research direction at last.

1. SUMMARIZE

Thanks to the fast development of various data acquisition methods, we can obtain all sorts of spatial data，especially the data of global remote sensing and mapping. In order to manage extract and analyse the spatial data much more effectively, it’s important to adopt new supervisor mode of global spatial data. And in order to effectively manage the extensive especially global multi-resolution geographical spatial data, in the 1980s last century many academicians put forward global discrete grid data model based on subdivision of regular polyhedron. The postulate is to split spherical surface into tiles with the approximately equal figure and area, using hierarchical recursive subdivision and corresponding address code of each tile instead of geographical coordinates to do all kinds of operation. Because the address codes not only show the position but also express the scale and precision, it has the potential ability to manage multiscale data. On the other hand, thanks to the especial mathematic characters of voronoi diagram, it becomes the most hopeful methods to solve dynamic GIS (Chen, 2002). And how to effectively combine the both becomes many academicians focus.

Voronoi diagram is a widely research issue in computational geometry, for its importance, it is an important geometry that is only inferior to convex hull. It has a lot of egregious funny mathematical characters, and it is a powerful tool to research and solve some issues of geoscience, computer science, mathematics and so on. However, because the earth is an approximate sphere, it has import significant to establish spherical voronoi diagram for the management of the global spatial data and dynamic maintenance of spherical spatial relation.

At present, it is less in research of the theoretics of spherical voronoi diagram abroad, but there are several typical algorithms among them, for example, Augenbaum (1985) gave the spherical voronoi diagram algorithm of *n* points using incremental method, and the time complexity is $O(n^2)$; Robert(1997) put forward the devider and conquer method to solve the generation of spherical voronoi diagram, and the time complexity is $O(nlogn)$; Zhao(2002) put forward the QTM-based algorithm for the generation of voronoi diagram for spherical objects, etc. Among the three algorithms, the first two methods are both vectorial algorithms aiming at spherical point set, and the last one aims at all kinds of different spherical aggregation based on specifically spherical triangular mesh. Until now, all the vectorial algorithms of spherical voronoi diagram are only effective to point set, and don't have operability for curve set and area set(Chen, 2002; Zhao, 2002). And the paper also adopts the clew of generating of voronoi diagram based on grid, which uses the spherical grid as the basic subdivision cell, adopts the transformation of grid interval, and then goes on with the research of spherical voronoi diagram, but we adopt hexagonal mesh instead of triangular mesh as the basic spherical subdivision cell. Compared with triangles, the hexagons have more advantages, for instance, they have more directions, provides the greatest angular resolution (Golay, 1969), have less interval of sampling, and all directions are the same, etc. So we put forward a spherical hexagonal grid

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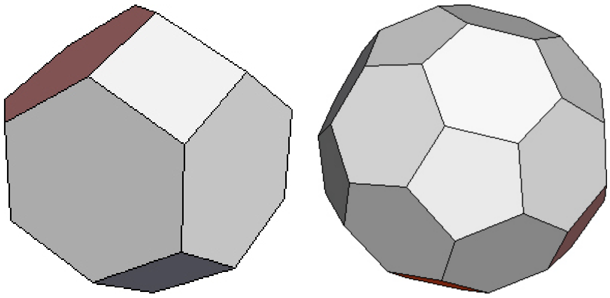
expression modes of different spherical entities and a corresponding algorithm of generating of voronoi diagram.

2. BASIC SUBDIVISION GRID AND THE DEFINE OF SPHERICAL VORONOI DIAGRAM

2.1 The Establish of Basic Subdivision Grid

The hexagon has received a great deal of attention as the basis for planar discrete grids. Among the three regular flat surface bedding polygons (triangles, squares, and hexagons), the hexagon is the most compact one. Because it can quantize the plane with the smallest average error (Conway and Sloane, 1988), and provide the greatest angular resolution (Golay, 1969). However we must pay attention to the fact that it is impossible to completely tile a sphere with hexagons. When a polyhedron is tiled with hexagon-subdivided triangle faces, a non-hexagon polygon will be formed at each of the polyhedron's vertices. The number of such polygons is in relationship with the number of polyhedron vertices, and has nothing to do with the grid resolution. For example, an octahedron has eight squares, while an icosahedron has 12 pentagons. They are all called Truncated Polyhedron, and also called Archimedean Solid, e.g. Figure 1.



(a) Truncated Octahedron (b) Truncated Icosahedron

Fig.1 Truncated Solid

Generally speaking, perfect entities with small faces will reduce the distortion, when they are transformed between the face of a polyhedron and the corresponding spherical surface (White et al. 1998). The tetrahedron and cube have the largest face size and compared with other entities, they are the mostly unlike a sphere ones. However, because the faces of a cube can be easily subdivided into square quadtrees, it was chosen as the basic platonic entity by Alborzi and Samet (2000). The icosahedron has the smallest face size, and as result the DGGSs defined on it tend to have the smallest distortions. So the icosahedron is the best choice for a basic platonic entity. Geodesic DGGSs, which is based on the icosahedron, include those of Williamson (1968), Sadournay et al. (1968), Baumgardner and Frederickson (1985), Sahr and White (1998), White et al. (1998), Fekete and Treinish (1990), Thuburn (1997), White (2000), Song et al. (2002), and Heikes and Randall (1995a, 1995b). And based on these consideration, we mainly discussed the icosahedron subdivision and hexagonal subdivision(also called Truncated Icosahedron Subdivision) in this paper.

How to project the discrete grid on spherical surface is also the problem that need to be discussed. And it has two clews, one is direct spherical subdivision, however the maximal bug of this method is that it is hard to define the subdivision, and the subdivision grids are hard to ensure the same areas, and the subdivision edges are hard to ensure that they're spherical major arc, so its precision is not high, and its operation is hard. The other one is projection method, which can make discrete grid map on the spherical surface. Compared with the last one, the operation of this method is good, and its deformation is controllable, while the mathematical reasoning is rigorous. So we adopt the second clew, and establish the direct relationship between the unfolded polyhedron and the spherical surface by ISEA Snyder equal area projection(Snyder, 1992), e.g. Figure 2. And then we extend it to multilayer subdivision grid. Owning to the restriction of the content, we will describe them in another article.

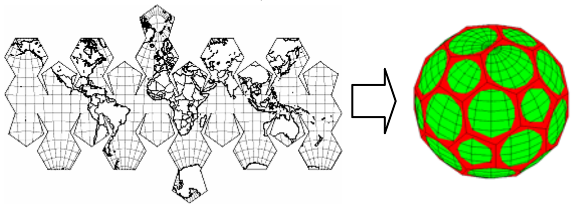


Fig.2 The Snyder Projection of Truncated Icosahedron

Based on the foregoing discussion, we choose the truncated icosahedron as the basic spherical subdivision cell, and use hexagonal grid to ulteriorly subdivide. And the concrete dispersed method is that its aperture is 4. Figure 3 is the result image of discrete grid map on spherical surface, which is projected by ISEA.

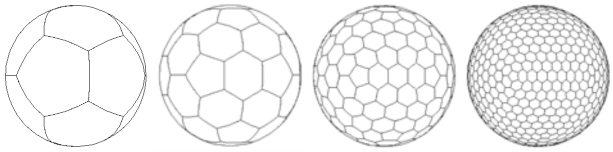


Fig.3 Spherical Hierarchical Subdivision Based on Truncated Icosahedron Using ISEA Projection

2.2 The Definition of Spherical Voronoi Diagram

Spherical surface *S* is not the Euclidian space, it is different from the plane, because they don't have the same embryo, so we must give the definion of spherical voronoi diagram before it is researched.

In order to express it more tersely, we establish the voronoi diagram on the surface of a unit sphere. Firstly, we define the voronoi diagram for a point set, given that point set $P=\{p_1, p_2, \cdots, p_n\}$ of *n* points on spherical surface, and for every point p_i in *P*, the locus of *x* is the one that all points closer to a point p_i in *P* than any another point in *P*.

$$\left\{ x \in S \,\middle|\, \left|\overset{\frown}{xp_i}\right| \le \left|\overset{\frown}{xp_j}\right|,\ j = 1,2,\cdots,n;\ i \neq j \right\} \qquad (1)$$

Among formula 1, $|x^\frown y|$ denote the nearest distance between *x* and *y* on *S*, i.e. the length of minor arc on spherical surface. If the central angle of $|x^\frown y|$ is τ, then $|x^\frown y|=1 \cdot \tau = \tau$. The polygon region $V(p_i)$ that satisfy formula 1 called the spherical voronoi region of p_i. The collection of all *n* voronoi region, one for each point in *P*, constitute the Spherical Nearest-point Voronoi Diagram of point set, or simply the Spherical Voronoi Diagram of point set.

And then we define the spherical voronoi diagram of arbitrary figure *G* on spherical surface *S*. If the figure *G* on spherical surface satisfies $G=\{g_i \in S|i \in N;\ i \geqslant 2\}$, then its spherical voronoi diagram can be expressed as follows.

$$V = \left\{ v \in S \middle| \exists g_m, g_n \in G; g_m \neq g_n; \forall g_i \in G, \left|v\overset{\frown}{g}_m\right| = \left|v\overset{\frown}{g}_n\right| \leq \left|v\overset{\frown}{g}_i\right| \right\}$$

The arbitrary figure G on spherical surface can be separated into point set, curve set and area set. The generating algorithm of spherical voronoi diagram for point set has already existed, and the generating algorithms of spherical voronoi diagram for curve set and area set are more or less difficult and juvenility, so we adopt the method of generate spherical voronoi diagram of different aggregates based on grid overlay. Because is has many advantages, for example it is brief in concept, well defined in hierarchy and easy to expansibility in high dimensional. And it gives a new clew to generate the spherical voronoi diagram. The paper adopts the generating of voronoi diagram based on spherical subdivision of truncated icosahedron hexagonal discrete global grid.

3. HIERARCHICAL MANAGE AND EXPRESSIVE MODE OF SPHERICAL ENTITY BASED ON TRUNCATED ICOSAHEDRON DGG

3.1 Multi-resolution Subdivision Method of Spherical Icosahedron

Spherical Subdivision was advanced by German cartological Fuller in 1944, when he was studying the map projection. And from then on, some academicians came to delve into different methods, for the sake of the management and analysis of global data. The paper adopts icosahedron as the basic spherical grid, and uses the method of Snyder equal area projection(ISEA) to get spherical surface map on plane, then uses hexagon whose aperture is 4 to subdivide every projection plane, and finally uses the inverse Snyder equal area projection to get planar grid map on the spherical surface. Figure 4 is the process.

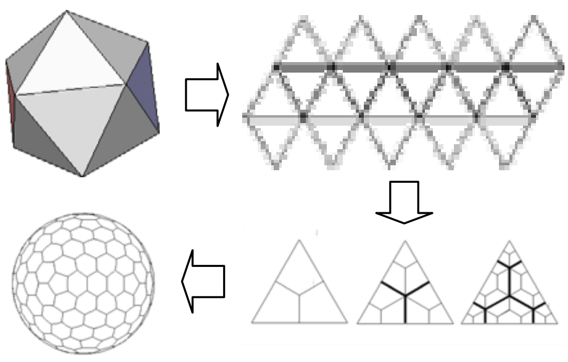
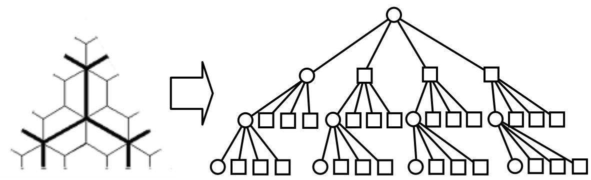


Fig.4 The Process of Spherical Hierarchical Subdivision

The idiographic multi-resolution subdivision is operated in process 3, but there is an issue, the use of multi-resolution, hexagon-based discrete grid systems has been hampered by the fact that congruent discrete grid systems cannot be built by using the hexagons; it is impossible to exactly decompose a hexagon into smaller hexagons (or, conversely, to aggregate small hexagons to form a larger one). The issue heavily confined the application of multi-resolution hexagonal grid. On the basis of the subdivision character of aperture 4, we put forward a new retrieval algorithm of multi-resolution grid nodes. Unlike the traditional retrieval of quad tree tile, this algorithm can retrieval and manage multi-resolution hexagonal grid more effectively. We will simply introduce the basic clew of the method.

3.2 Management of Three Leaves Nodes and Encoding of Tiles

Although the discrete grid cannot use the managerial method of hierarchical tiles because it adopts the hexagonal subdivision mode on triangular plane whose aperture is 4. However, if examine the mode more carefully, we can find that all nodes of the last hierarchy are kept down in the next hierarchy, and there are three new nodes pushed out based on the old ones, and the new nodes can compose a new structure called Three Leaves Node Structure, e.g. Figure 5.a. At last, the three new nodes and the last hierarchical old node can make up a new node quad tree, Figure 5.b. And then, we can manage the hexagonal tile data via managing the leaves nodes of quad tree.



a. Three Leaves Structure b. Quadtree Manage

Fig.5 The Quadtree Manage of Three Leaves Nodes
(○: Reserved Nodes, □: New Nodes)

For an arbitrary point P on the spherical surface, it is mapped on one of the triangular plane. And for the N nodes of arbitrary subdivision hierarchy n, the point must be in one of the some one's (L_i) region, i.e. formula 2.

$$|PL_i| \leq |PL_j| \quad j = 1,2,3,\cdots,N; i \neq j \qquad (2)$$

From formula 2, we can fine that the aggregate of all P points composes the voronoi region of the node L_i, and the collection of all N voronoi regions, one for each point L_i in L, constitutes the voronoi diagram of point set L, e.g. Figure 6.

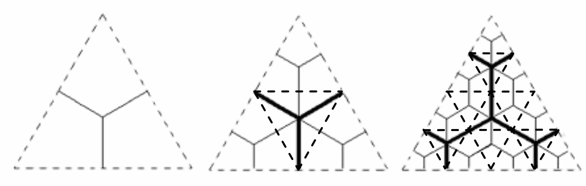


Fig.6 The Voronoi Diagram of Different Hierarchical Nodes (Dashed Line Represent Voronoi Diagram)

The voronoi diagram of nodes constitutes the overlay of the regular triangles on single triangular plane, just the same as traditional QTM(Quaternary Triangular Mesh) subdivision. In order to accomplish the quick indexing and management of the hexagonal tile, we manage the hexagonal tiles through the assistant of node quad tree, so we need to manage nodes through encode. Because the voronoi diagram of node set made up QTM subdivision, we note the coordinates of nodes through typical QTM coding mode. And the detail that we can find the detail in literature(Zhao, 2003). Figure 7 is the order of QTM

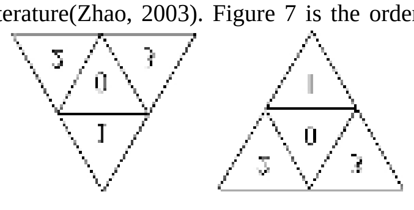


Fig.7 Order of the QTM Code

coding sequence in different directions.

For an arbitrary point on spherical surface, how can we switch it into the voronoi diagram space of node set is an important issue, so we put forward the Three Orientation Translating Algorithm and establish the three orientation geographic grid coordination, and we will introduce this algorithm in another article. We can switch an arbitrary point on spherical surface to the voronoi diagram space of node set through Three Qrientation Translating Algorithm, and establish the code indexs of nodes, its form is the same as the code indexs of QTM tree structure, i.e. $a_0a_1 \cdots a_n$, among them, a_0 stands for the basic subdivision hierarchy, while for icosahedron, i.e. $a_0 \in [00,19]$, *n* is the number of subdivision hierarchy, $a_1 \rightarrow a_n$ is quaternary code(0-3) and the maximize of *n* is 30(Zhao, 2003). Because encoding above only accomplishes the coding of nodes of an arbitrary point, we must add one identification code to the above codes and then we can accomplish conversion between hexagonal tile and spherical point, i.e. $a_0a_1 \cdots a_nb$, $b \in$ [1, 3]. The identification code *b* shows the spherical arbitrary point belongs to which hexagonal tile. Just as it is showed in Figure 8.

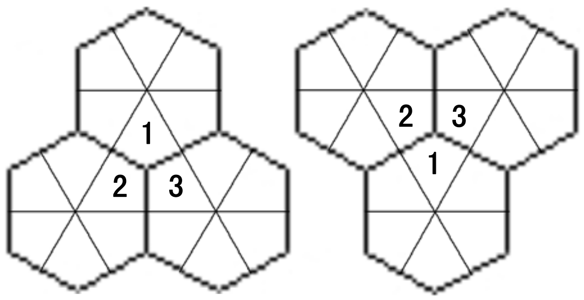


Fig.8 The Order of Identification Code

We can confirm the hexagonal tile's position of spherical arbitrary point through identification code, it means a hexagonal tile is divided into six regular small tiles if we do these. For the points on different small tiles of the same hexagon, although their codes are different, they belong to the same hexagonal tile. We adopt a new quick indexical method to search the codes of six small tiles which are generated by the same node, and solve the issue in a better way. Also, if we adopt this indexical method, it is a better choice to solve another problem of tile splice of different hierarchy. And we will describe them in another article. Figure 9 shows the hexagonal tile which is made up of six small tiles(the number in the figure is identification

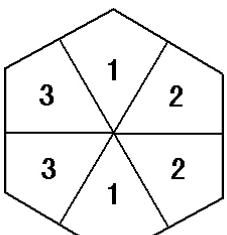


Fig.9 Conjunct Image of Hexagonal Tile

code).

3.3 Expression Mode of Spherical Entity Based on Hexagonal Grid

In order to operate spherical entity more simply, the mode must be not only standard, but also be of great advantage to design database of different resolution spatial data and dynamic operation of local data. In this paper, we combine the spherical entity with spherical overlay, and use the code of hexagonal tile to manage spherical point set , curve set and area set.
Point set: It is easier to express the point set by an hexagonal address code, however, the address code of hexagon is not unique, which is made up of six small tiles. The coordinate of the point must be in one tiles, so the hexagonal address code of the point is specified by the address code of the small tile. In this way, we can search the other five express modes through a new quick indexical method, and the different expression modes stand for different directions. Although it may be not very useful in the dynamic analysis, it is very useful to record the points' directions, and the analysis will be much more easy and effective.

Curve set: Spherical curve entity is expressed by connecting of a series of points, just the same as the point set. And the coordinate is expressed by the hexagonal address code. For one point in an arbitrary curve, it has two different directions i.e. spherical normal direction and spherical tangent direction. We adopt two different encoding modes to record spherical curve entity and mark its directions, i.e.

①Because in the analysis of GIS the tangent direction of every point in the curve is very important, especially in the process of converting raster data format into vector data format, such as the process of extracting the line symbol and sampling the point vectorization. So the direction of point become very important. Owning to the limitation of the traditional vector and raster data mode, many scholars use chain-coding(Freeman, 1977), curvature scale(Bober 2001) etc to record the tangent direction of the points in a curve as the assistant tool. But in this way, it will be more complex and unefficient .This method increase complexity and decrease efficiency. Besides, it cannot record the coordinate and direction at one time;

② In some especial domain, such as the spherical voronoi diagram generating process, which will be mentioned in the following text. It is more important to record the spherical normal direction of a curve. So we use small tiles' normal direction code instead of the hexagonal tiles' address code, and show the dilative direction of every point. It is propitious to dispose later.

Each of the two expression modes of spherical curve set has their own use. And we can also constitute different encoding modes based on specific demands. All these different encoding modes can be quickly switched by the indexical method. We will discuss the issue another article.

Area set: Spherical area entity is represented as a clockwise surrounding boundary line and a series of hexagonal cell address codes. Compared with the traditional vector data expression, it is easy to infer spatial relationship. This kind of method to express the two dimension entity not only possess two advantages of traditional surrounding boundary pattern and regular grid pattern of manifestation, but also abide spherical geometric character of digital area(Zhao, 2002). The boundary of area adopt the encoded mode of curve entity, choose the different direction according to specific purpose, and the paper adopt encoded mode ② , the encoded mode inside adopt the encoded mode of point set.

4. GENERATION AND ANALYSIS OF SPHERICAL VORONOI DIAGRAM BASED ON HEXAGONAL DGG

4.1 Generation Algorithm of Spherical Voronoi Diagram Based on Hexagonal Grid

The generating clew of spherical voronoi diagram based on hexagonal grid according to the elements of regional dilation in mathematical morphology(Chen, 2002; Zhao, 2002; Chui, 2002). Firstly, define the two basic operations of spherical hexagon: if A is primary region of spherical hexagonal grid, and B is structural element, Figure 10 shows the operating definition of erosion and dilation on spherical hexagonal grid.

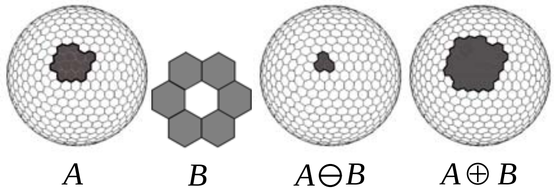


Fig.10 Erosion and Dilation Operation Based on Spherical Hexagonal Grid

The concrete algorithm of generating the spherical voronoi diagram is shown as follows:

(1) Firstly, extract the spatial vector data from the spherical

$$\left.\begin{array}{ll}\text{Erosion} & A\Theta B = \cup b \in BA_b \\ \text{Dilation} & A \oplus B = \cap b \in BA_b\end{array}\right\} \quad (3)$$

surface, and then convert them into address codes. Regardless of the points, lines or the areas, the ultima operation is an aggregate of lines. In this case, we adopt the second encoding method to record the lines' normal direction.

(2) By using the normal directions which we have just recorded in the first step, we can easily find the address codes of the hexagon tiles in the next grads direction. Actually, these codes are just different in the identification codes. So what we should do next is to convert them into the normal direction codes.

(3) Delete the hexagon tiles which have the same codes, and then a new spherical surface and a new verge will engender by inflating or decaying. And the new verge have the same distance with the original one.

(4) Repeat the steps above until the inflation(or decay) area of different spherical surface entities intersect the others. And this step will stop until the codes (except the identification codes) of different entities become the same or different codes belong to the same hexagon tile.

(5) Repeat the steps from (1) to (4), until the whole sphere have been completely searched and the intersectant hexagon tiles engender a new verge. And then use the first method to generate the spherical voronoi diagram.

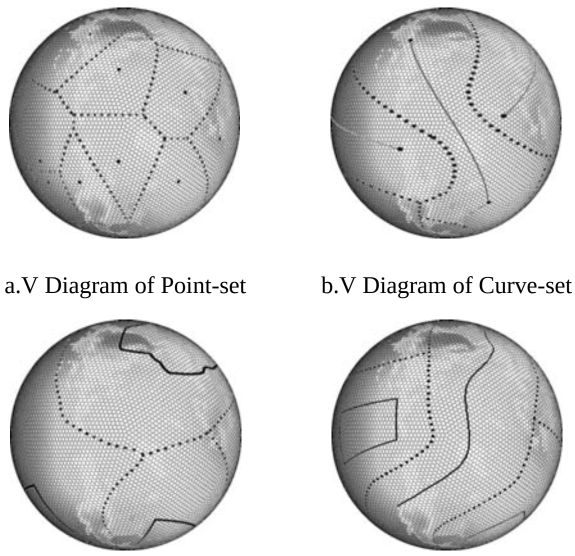
4.2 Experiment and Analysis of the Generation Algorithm of Spherical Voronoi Diagram Based on Hexagonal Grid

Based on the principles above, we developed a relevant system on the three-dimensional visual platform by using OpenGL. And the computer configuration is: PIV 1.5GHz, memory 512MB. And we adopted hexagonal tiles address coding method to calculate the voronoi diagram of different entities. In the experiments, we used mixed collection, and it is shown in Table 1. The level n equal to 6 (n=6). The experiment result is shown in Figure 11. And the experiments proved that the spherical voronoi diagram based on hexagonal grid satisfies the precision of the theoretic spherical voronoi diagram. The consumed time is related with the initial aggregate and the levels of the grid. So under the same precision, we can adjust the levels of grid to control the consumed time.

Table 1：Experimental Data

	Points	Lines	Areas
Point V Diagram	19	0	0
Line V Diagram	0	4	0
Area V Diagram	0	0	5
Mixed V Diagram	1	1	7

As to algorithm of spherical Voronoi Diagram based on hexagonal grid, the concept is easy to understand; the structure is simple and the levels is legible. After been operated, the calculation result can construct the dynamic data format.



a.V Diagram of Point-set　b.V Diagram of Curve-set

c.V Diagram of Area-set　d.V Diagram of Composite object sets

Fig.11 The Voronoi Diagram of Spherical Different sets Based on Hexagonal Grid

Compared with the generation algorithm of Voronoi Diagram based on QTM triangular grid, this kind of algorithm has a lot of advantages.

1) First of all, thanks to the isotropy of the hexagonal grid, the inflation (or decay) result is more accordant to the ideal model, and it is easy to operate.

2) Secondly, by using Snyder equal area projection, all the hexagonal tiles have the equal area. And by using the morphologic mathematics operation method, the result is more accurate. Theoretically, the error of the voronoi diagram can be strictly controlled in a hexagonal grid.

In order to analyse the influence of the hexagonal grid of different size to the generating rate, we adopt the mixed aggregate(e.g. 1 point, 1 arc, 7close curved face) to operate at different hexagon levels. Among them, the subdivision hierarchy *n* choose 3、4、5、6、7, the spherical voronoi diagram of different subdivision hierarchy is showed in Figure 12, take three hierarchies as an example, the consumed time is shown in Figure 13.

a. Hierarchy 5 b. Hierarchy 6 c. Hierarchy 7

Fig.12 The Voronoi Diagram of Different Hierarchical the Same Sets(Real line and isolated point are original sets, dashed is generating spherical voronoi diagram)

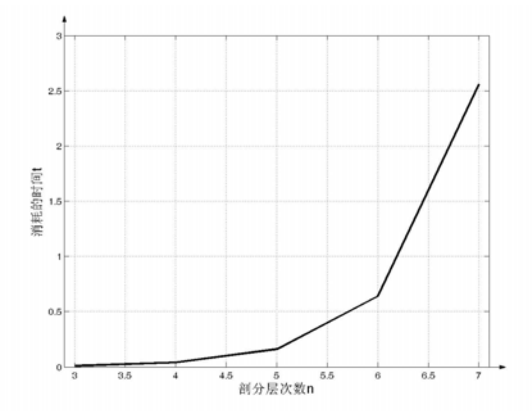


Fig.13 The Time of Diffenent Hierarchical Subdivision Generate Voronoi Diagram

The dynamic stability of the voronoi diagram makes it become a useful tool to solve the global dynamic data model. In this paper, it gives the generating algorithm of spherical voronoi diagram of different sets based on truncated icosahedron hexagonal discrete global grid. After analyzed the experiment results, we made the following conclusions:

1. The concept of the algorithm of spherical voronoi diagram based on hexagonal grid is very intuitionistic. Besides, it is easy to operate and extend. Regardless the points, lines or areas, it consumes the same time to calculate them.

2. The calculating time of the algorithm is mainly consumed at the choice of subdivision hierarchy. With the increase of the hier-archy, the consumed time takes on an exponential increase(Figure 13). For the low precision spherical surface data, we can adopt lower subdivision grid hierarchy to reduce the calculating time. Besides, the precision can satisfy the requirement. For the high precision spherical surface data, we can adopt multilayer hexagonal grid structure that is from high resolution to low resolution to approximate gradually spherical voronoi diagram of different sets.

3. When we operate with the data of large quantity, including the data with complex original aggregates and those with excessive levels, the algorithm mentioned in this paper have some localizations. It is mainly because that, with the increase of the scale of the aggregates, the complexity of the consumed time takes on an exponential increase accordingly. It is disadvantage to generate the spherical voronoi diagram of large quantity data. And it is what we will study next.

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