

FINITE DIFFERENCE METHOD RUNOFF MODELLING USING VORONOI CELLS

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ABSTRACT:

Finite difference runoff modelling has usually been done using a regular grid representation of the terrain. This has various disadvantages, as the regular pattern does not conform well to observed features such as watersheds, the runoff pattern is biased to the grid axes, and original data points are lost. We propose a flow modelling method using Voronoi cells associated with TIN models. A random Voronoi pattern is added to the original data. This avoids the issues of grid based methods, as there is no axis bias, points may be added anywhere and original data points may be retained. Our flow model simply requires a set of “buckets” to hold the water (the Voronoi cells) and slope information to provide the local runoff rate (the Delaunay edges).

1. INTRODUCTION

The hydrologic cycle is the circulation of water within the earth's hydrosphere. It describes the continuous exchange of water between atmosphere, land, surface and subsurface. One of the important components of this cycle is surface runoff. It is water, from rain, snowmelt, or other sources, that flows over the land surface. It occurs when the intensity of precipitation exceeds the ground's infiltration rate. Once the precipitation rate exceeds the infiltration rate of the ground, the soils become saturated and all depressions begin to fill up with water. If the precipitation continues the excess of water above the soil starts to move downstream. Elements affecting runoff include precipitation factors, like its nature (rain, snow, ice, etc.), intensity, spatial distribution, and surface characteristics, including its topography, interception rate, storage capacity and porosity.

There are many commercial and academic systems allowing digital runoff simulation. They consist of two main components: the digital terrain representation and the hydrologic processes governing overland water flow.

The two most popular digital representations of terrain are the regular grid and the triangular irregular network (TIN), both suitable for dynamic runoff simulation. The availability and simplicity of grid models, which are matrix data structures with a topographic elevation value in each matrix node, made them the most popular representation and base for various hydrological modelling processes. However grid based terrains suffer from many disadvantages. They are usually created by interpolation of the original data, which are elevations sampled at discrete points or contour lines. Often the quality of grid terrain models is poor and they do not reproduce the shape of the original surface. Their constant spacing requires high resolution to preserve details of the surface (which means high storage cost). TIN models, usually represented by the Delaunay triangulation, have many advantages: their resolution adapts to the data distribution and they conform to the original data. In case of terrain construction from contour lines some additional processing of resulting TIN models is necessary to insert peaks, pits, ridges and valleys.

The two most widely used formulations for water flow modelling are The Finite Element Method (FEM) and the Finite Difference Method (FDM). FEM methods use irregular meshes, FDM methods are often grid based. A form of the FDM, the Integrated Finite Difference Method (IFDM), is based on non-structured meshes. A manual method of irregular cells construction originally suggested by (MacNeal, 1952) was fully defined by (Narasimhan and Witherspoon, 1976), and was automated using the Voronoi diagram by (Lardin, 1999), showing that an iterative finite difference scheme could be developed for Voronoi cells, using the dual Delaunay triangulation. The volume of water moved between adjacent cells depended on the gradients of the triangle edges.

An important component of hydrologic models is the routing of flow across a terrain surface. In many grid based runoff models directions are restricted to 45 degree direction increments. In the IFDM method flow is distributed irregularly, based on the distribution of cells adjacent to the processed cell.

We propose a runoff modelling solution using the IFDM method for terrain models created from contour lines, based on the Voronoi diagram generated from the Delaunay triangulation.

2. TIN GENERATION

The terrain represented as a TIN model is generated using a simple incremental algorithm for point insertion (Fig 1b). The data comes by scanning contour maps (Fig. 1a) or from available digital datasets, for example in SHP format.

In case of TINs created from contour lines flat triangles (Fig 2a), connecting vertices with the same elevation, are removed by enriching the model using the skeleton method (Dakowicz and Gold, 2003). Insertion of skeleton points reconstructs missing features, like summits, valleys or ridges (Fig 2b).

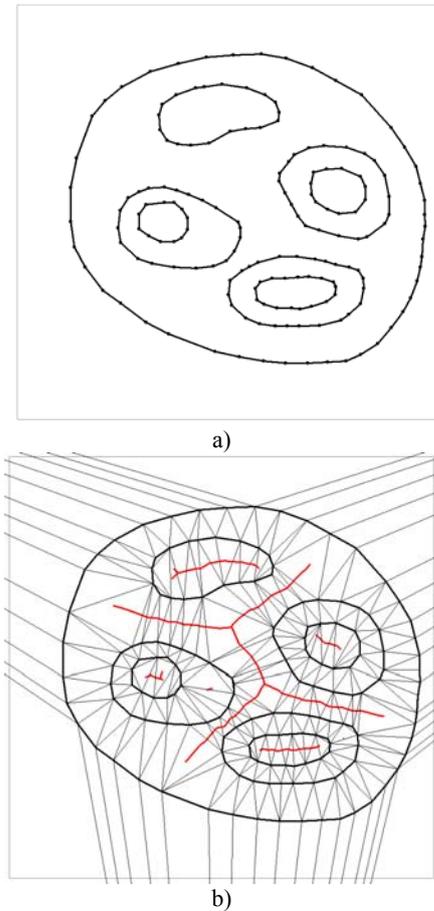


Figure 1. The initial model; a) the contour data; b) the initial TIN with draped skeleton branches

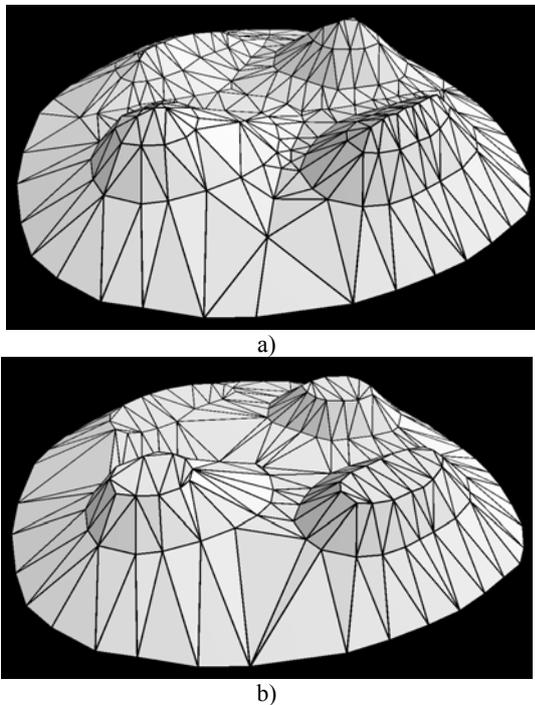


Figure 2. The 3D view of TIN models; a) the initial TIN with flat triangles; b) the enriched TIN with flat triangles removed

The distribution of nodes in the resulting TIN model is irregular – they are concentrated along the original contour lines and skeleton branches used for the enrichment, with large empty areas in between. The efficiency and stability of IFDM simulation is determined by the shape of cells. The method is based on the idea of moving water volumes between neighbouring Voronoi cells with the amounts determined mostly by the gradient of the cells and the width of the common edge. A problem occurs when there is a large elevation difference between two cells sharing a very narrow common Voronoi edge and when there is another neighbouring cell with higher elevation sharing a wide common Voronoi edge. Then a large amount of water accumulates in that cell – waters flows in very quickly through wide common edge and leaves it slowly through this narrow common edge, causing an unnatural behaviour of the simulation. To avoid such cases, which mostly occur in sparse models, it is recommended to regularize the shape of the cells by thickening the mesh data with additional nodes.

3. FLOW FRAME

There are several actions needed to make the cells similar in size. The first is to add a rectangular frame around the mesh, bounding all data points and separating the data from the big triangle bounding the Delaunay triangulation. The size of each Voronoi cell associated with the big triangle vertices is infinitely large. The areas of Voronoi cells of the nodes directly connected to the big triangle vertices are significantly larger than the areas of the cells inside the mesh. Adding a frame removes those large cells.

The frame is defined by four corners, each with a specified elevation value. The corners are linked by strings of nodes spaced by the disk size value and with linearly assigned elevation values (Fig. 3). Elevation values of the frame corners are arbitrary and can be specified by the user. Elevation values for the rest of the frame nodes are computed using the elevations of the corners. All the frame nodes are marked with ‘no flow’ flags and water does not flow along edges connected to those nodes. We currently use a rectangular frame, but the shape of the frame can be arbitrary.

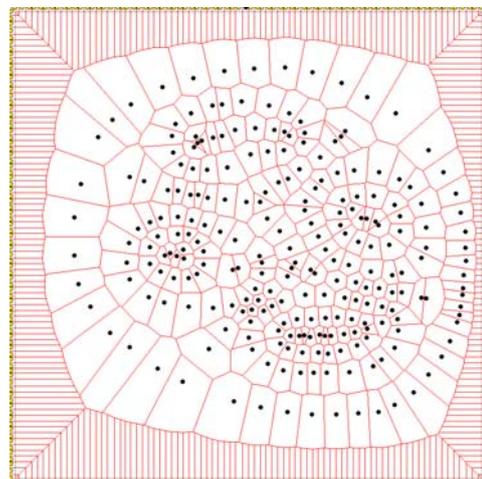
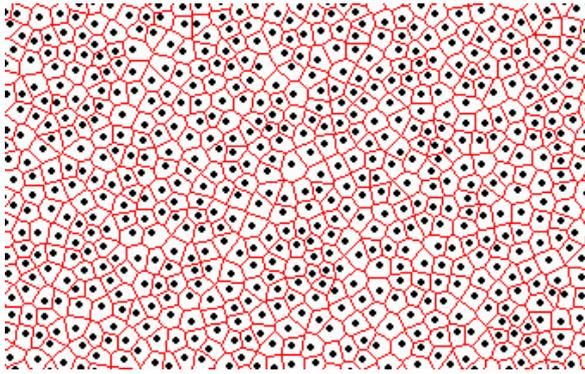


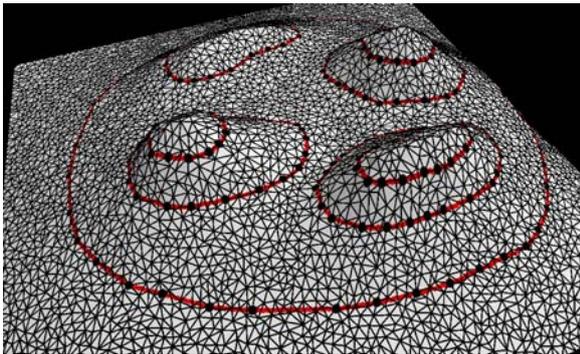
Figure 3. The Voronoi Diagram of the enriched TIN model with the flow frame

4. MODEL ENHANCEMENT

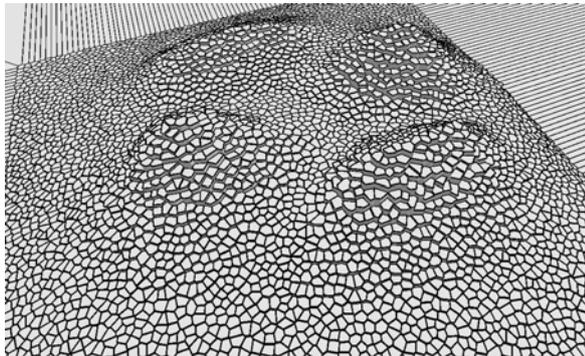
The second action is to thicken the data inside the frame. We generate a random pattern of points and insert them into the existing TIN with a guaranteed minimum spacing (Fig. 4a). Each additional point with specified x and y coordinates has the elevation value assigned using the Sibson interpolation method (Fig. 4b and 4c) (Sibson, 1982; Gold, 1989). Other interpolation methods leading to a natural and smooth surface can be used. A relatively large disk radius assigned to points and used for collision detection prevents insertion of points closer to each other than a doubled radius of the disk. That leads to a fairly regular distribution of the triangulation nodes.



a)



b)



c)

Figure 4. The enhanced TIN, a) a Voronoi diagram; b) a 3D view of the Delaunay triangulation with superimposed contour lines; c) a 3D view of Voronoi cells

5. MODEL INITIALIZATION

The runoff simulation process consists of two main steps – the initialization and the iterative processing of the model. In the real world the initialization step corresponds to rainfall over a specified area, so at a specified moment a certain amount of water covers the surface.

In our process Voronoi cells are treated like “buckets” and water is “poured” into each of them (Fig 5a) by scanning the whole map and assigning a specified water height value to each node, which is stored in an auxiliary field, along with the coordinates of the node. As a result of this step all the cells are filled with water, the level of water is the same in each cell and the water volume in each cell can easily be computed using the cell’s area and the current water height.

6. FLOW SIMULATION

The next step is the flow simulation itself, performed iteratively for the whole map for a number of iterations specified by the user. In each iteration each cell is examined and water is moved to its neighbours as required. Firstly the water volumes to be moved between adjacent cells have to be computed. All edges of the mesh are processed and each edge is used once. One cell is a source of water for all the neighbouring cells with lower elevation and at the same time it receives water from the higher neighbouring cells. The elevation values are combined with the water level in the cell – that solves the “pit” problem in runoff modelling

In the case of cell p surrounded by i neighbours, having each common edge perpendicular to the edge connecting their centres, the flow equation for groundwater flow is as follows (Narasimhan and Witherspoon, 1976):

$$\sum_{i=1}^n K_{ip} \frac{(h_i - h_p)}{x_{ip}} * F_{ip} = V_p S_{sp} \frac{dh_p}{dt} - V_p R_{sp} \quad (\text{eq. 1})$$

K_{ip} – average hydraulic conductivity between cells i and p

x_{ip} – distance between the centres of cells i and p

F_{ip} – contact surface between cells

V_p – saturated volume of cell p

S_{sp} – storage coefficient of cell p

R_{sp} – volume loss

h_i, h_p – water height for cells i and p

$\frac{dh_p}{dt}$ – rate of change of water height in cell p with time

For surface flow F_{ip} = length of the contact edge, V_p = cell p area * h_p . ($S_{sp}=1$ and $R_{sp}=0$ are ignored)

A cell with a higher elevation at one end of the edge is set as the source and a specified amount of water flows from this cell to the cell at the other end. The volume of this flow depends on

the source cell area, the gradient $\frac{h_i - h_p}{x_{ip}}$ of those two

neighbouring cells and the length F_{ip} of the common Voronoi edge. Gradients are computed using the distance between two nodes and their elevation values combined with the level of

water. Two other factors affecting the volume are the time interval T between iterations and the resistance K_{ip} of the surface at each node location. The time parameter is set before starting the process. The resistance depends on the surface material and it can be different in various parts of the map. In our preliminary model the same resistance value is set for the whole map, but it changes according to the current level of the water in the cell – an increase of the water level decreases the resistance, which means that deep water flows faster than shallow. In practice, before applying the resistance parameter we test if the water level in the cell is higher than the specified threshold value – in such case a predefined smaller resistance value is used.

If we process only outflowing water from cell p (incoming water will be calculated for the other cell i) then the water volume lost from cell p to cell i is

$$VOLM = \Delta t K_{ip} \frac{h_i - h_p}{x_{ip}} F_{ip}$$

for a time step Δt . The time

and resistance parameters can be adjusted by the user at any time during the process.

While processing the whole map the VOLM values are stored in each cell as the current volume difference dVOL, which can be a positive or negative number. At the beginning of each iteration this value is set to zero in every cell and then updated when any of the adjacent edges is processed. After completing the map processing the dVOL value is used to update the water level in each cell. The VOLM calculated for an edge contributes to dVOL values of its both endpoint cells, increasing the volume for the lower node and decreasing for the higher. For example if a cell has five neighbours and two of them are above and three of them are below, then the VOLM will be computed for each pair of cells and the final dVOL value in this cell will be a result of transporting water to two lower cells and receiving it from three upper cells.

After processing all the cells the whole map is processed again and the computed volumes of water dVOL are used to determine new water levels (Fig. 5b). When the dVOL is positive then the water level increases, when it is negative the level decreases. The whole process is repeated for a user-specified number of iterations (Fig. 5c-5d).

We illustrate an imaginary case of a river watershed bounded by a no-flow frame in Fig. 6a. The river has a constant gradient and the outlet is the lowest point of the model. The outlet area was processed, adding horizontal walls connecting the real watershed with the flow frame to avoid water flowing sideways. In the initialization step a specified amount of water is added to the model, so each cell is covered by water with equal height (Fig. 6b). After starting the flow process water flows downwards, accumulates in the river bed and runs downwards towards the outlet (Fig. 6c-6e).

If the outlet is closed by no-flow nodes then water accumulates in that area (Fig. 7a) and its height increases after each iteration until the level of water in the whole watershed is even. If the outlet is opened by marking nodes with constant head flags, so they absorb water only and their heights do not change, then water leaves the watershed (Fig. 7b) and the total volume of the water in the model decreases after each iteration, until there is no water left within the watershed.

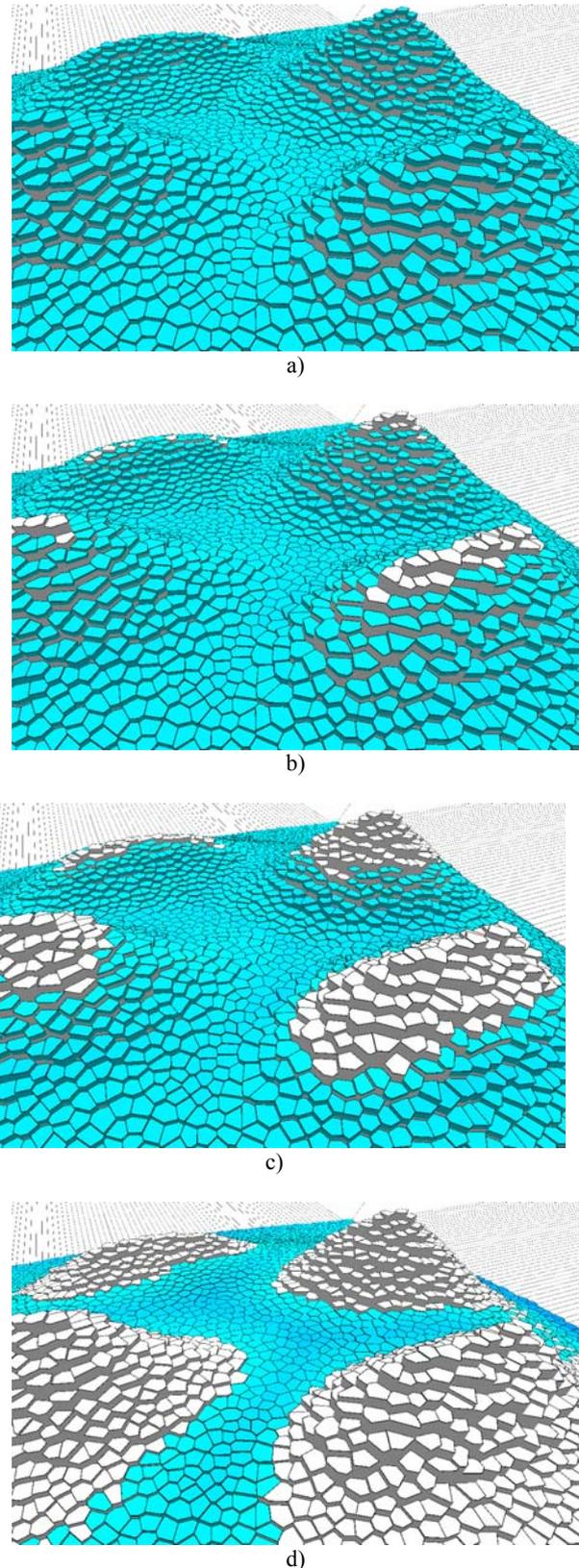
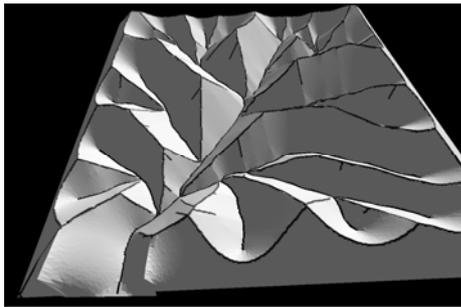
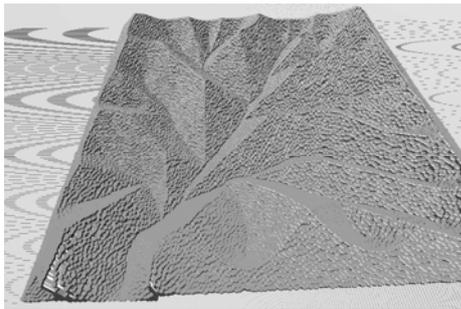


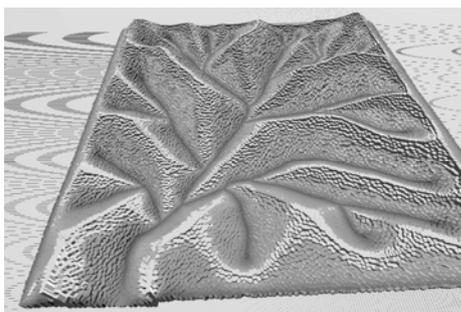
Figure 5. The 3D view of the flow model; a) the initial model; b-d) the model after various numbers of iterations



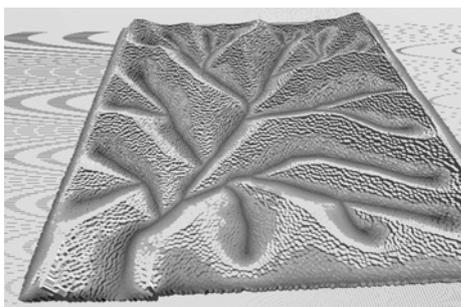
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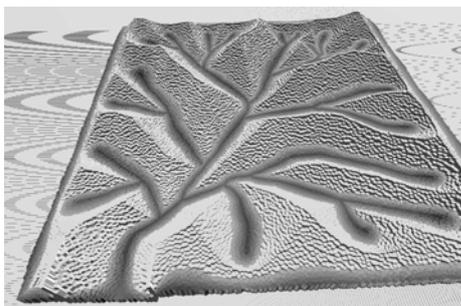
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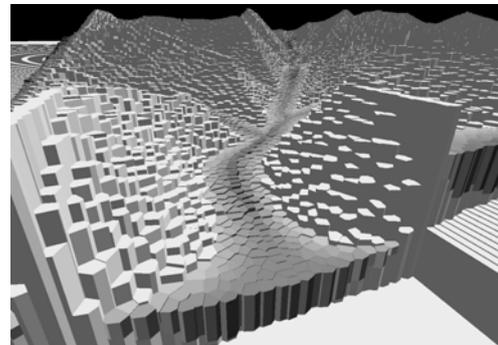


d)

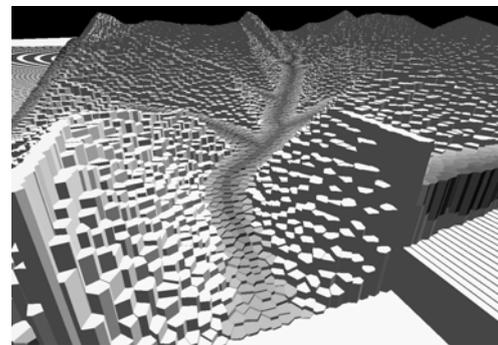


e)

Figure 6. Flow simulation using river data; a) TIN of the enriched river data; b) Voronoi cells with initialized flow process; c-e) flow model after various iterations of the process



a)



b)

Figure 7. River outlet after 250 iterations; a) blocked with accumulated water; b) open using constant head points

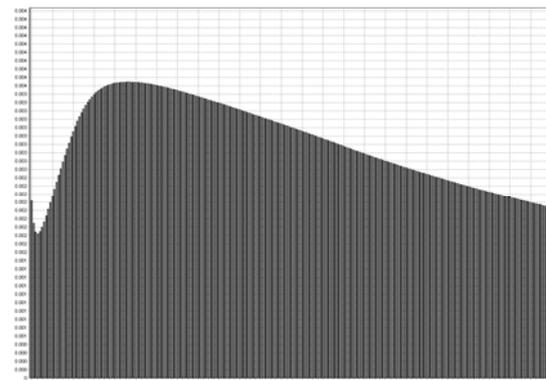


Figure 8. Water volume histogram for the river outlet after 250 iterations

7. CONCLUSION

We have proposed a flow modelling solution for Voronoi based terrain models using the IFDM method. The process has been implemented, tested on various data and proved to work well. A visual inspection shows a reasonable distribution of the water in the model at each step of the process. Water levels have been monitored at specified locations and the shape of the hydrographs conforms approximately to the theoretical assumptions (Fig. 8). This shows the water volume at some

point at the river outlet, as it changes over time. As expected, the flow volume peaks soon after the rainfall (our initialisation) and decreases steadily thereafter. The main drawback of the method is its speed - mostly due to the complexity of terrain models.

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