A MODEL FOR TOPOLOGICAL RELATIONSHIPS BASED ON EULER NUMBER

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ABSTRACT:

Topological Relationships between spatial objects is a very important topic for spatial data organization, reasoning, query, analysis and updating in Geographic Information Systems (GIS). The most popular models in current use have fundamental deficiency. In this paper, a whole-based approach is pursued to model binary topological relationships between spatial objects, in which (i) a spatial object is treated as a whole , (ii) appropriate operators from set operators (intersection, difference) are selected to distinguish the topological relations between spatial objects; (iii) three types of topological invariants are used for the computational results of set operations – contents, dimension and Euler-number. This approach overcomes the shortcoming of current models.

1. INTRODUCTION

Topological Relationships between spatial objects is a very important topic for spatio-temporal reasoning, query, analysis and updating in Geographical Information Systems (GIS). Since then, many papers on this topic have been published by researchs from the computing science and GIS communities. Many models of spatial topological relations have been proposed in the last decade, these models can be classified into 2 categories, i.e. decomposition-based and whole-based (CHEN Jun & ZHAO Renliang, 1999, Li Zhinlin, etc, 2002).

The former includes 4-intersection model (Egenhofer，Franzosa，1991), 9-intersection model (Egenhofer,1993) and the extensional models (Clementini E. et al, 1994; Clementiniand & Di Felice,1995), Voronoi-based 9-intersection model (CHEN Jun,etc，2001). There are many imperfections associated with the decomposition-based approaches mentioned above, the fundamental deficiency associated with the decomposition-based models is the inconsistency in the definition of line object, as pointed out by LI Zhilin et al.

In the category of whole-based approach, the main work includes the spatial logic model (Cohn et al. 1998) and voronoi-based spatial algebra (Li Zhinlin, etc, 2002). The former is based on “region connection” ontology, and unfortunateky, it’s distinguish ability is limit, many relations can not be distinguished by this model. The latter is based on voronoi region, which treats a spatial object as a whole, use the results of set operations and the object’s voronoi region to distinguish the topological relations between spatial objects. The main advantage of such work is that it introduces more set operations, which enhanced the distinguish ability of whole-based models.

However in voronoi-based spatial algebra, many topological relations’ distinction depends on the spatial objects’ voronoi region, as shows in section 2. As Spatial object’s voronoi region essentially reflects the influence range (orbit) of discrete object at the same time. It can’t be used to describe the topological relationshipof associated tessellation objects and the objects at different time, which is very important in the updating process and spatio-temporal reasoning. In order to overcome the deficiency of whole-based models without voronoi region, an alternative approach based on Euler-number is proposed in this paper. This is a whole-based approach. In order to capture the results ’difference in detail of the set operators, Euler-number is introduced.

Following this introduction is a review and analysis of the spatial algebra based on voronoi region, the fundamental deficiencies and the effects of set operators are examined, the strateges of this study will be proposed in section 2. A whole-based spatial algebra for topological relations based on intersection and difference operators is presented in section 3. A spatial algebra for topological relations based on Euler-number is proposed in section 4. The possible topological relations using this modified model are discussed in section 5. The summary is given in Section 6.

2. THE STRATEGIES USED IN THIS STUDY

In order to develop an appropriate strategy for this study, a critical examination of existing models is necessary. From the analysis of existing literature, an observation arises, the topological relationship model developed from decomposition-based to whole-based, from single value (content) to several values (content, dimension and number of connectd components), from single set operator (intersection) to several operators. The voronoi-based spatial algebra is a whole-based approach, which uses several values (content, dimension and number of connectd components) and several set operators, such as intersection, union, difference, difference by, and symetric difference. It is the most advisable approach in the existing models.

2.1 The Insufficiency of Voronoi-based Spatial Algebra

In the voronoi-based spatial algebra, the topological relations between object A and object B can be described by EQ (1):

$$R(A,B)=\begin{pmatrix} A\varpi B & A\varpi B^{V} \\ A^{V}\varpi B & A^{V}\varpi B^{V} \end{pmatrix} \tag{1}$$

where ω denotes the set operators, ω= {∩, ∪, /, \, Δ}, Av, Bv denotes the voronoi region of spatial object A and B respectively. The values of set oprator include “content”, “dimension”, and “number of connected components”.

In the spatial algebra, voronoi region of spatial objects play a very important rols, without voronoi region some topological relations, i.e. “cover” and “contain”, “covered by” and “inside” between 2-dimensional objects will be confused, as table 1shows.

Spatial object’s voronoi region essentially reflects the influence range (orbit) of discrete object at the same time. There are some insufficiency using it to describe topological relations between spatial objects.

1) It is not fit for the tessellation area objects. For example, parcel is not discrete object. In theory, all parcels at the same time form a complete coverage of space (except streets). Such object’s voronoi region is itself. The EQ (1) of voronoi-based spatial algebra changes to EQ (2), “cover” and “contain”, “coveredby” and “inside”etc. relations between such objects can not be distinguished by the voronoi-based spatial algebra.

$$VW(A,B)=A\varpi B \tag{2}$$

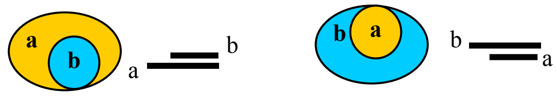
2) Spatial object’s voronoi region usually used to reflect the influence range (orbit) of objects at the same time. Objects at different time can not be used to partition the same space. Usually, people don’t discuss the the influence range (orbit) of objects at the different time. So voronoi region does not be used to describe the topological relations at different time. However in updating, topological relationship between parcels (boundary lines) before and after change provides the foundation for the determination of entities’ change type.

In addition, the voronoi-based spatial algebra is a generic model for spatial relation, as analyzed above, the voronoi region in this model does not fit to all circumstance. Another observation arised, whether the five set operators are neccessory in the description of topological relations? So the effects of the five set operators in topological relations’distinguish will be discussed in the next session.

2.2 The Effects of Set Operators

In voronoi-based spatial algebra, five set operators, i.e. intersection (∩), union (∪), difference (/), difference by (\), and symetric difference (Δ) are utilized to distinguish spatial relations. Now let us analyze the effects of these set operators in the description of topological relations.

1) The Effects of Difference (\)and Difference by (/)
When the result of intersection between spatial object a and b is not empty, the result of the differrence (or difference by) can reflect the difference in size between object a and b. As Fig.1 shows, if a covers b, then a\b=-∅, a/b=∅; a covered by b, a \ b=∅, a / b=-∅. So difference (\) and difference by (/) play an important role in the description of cover/ coveredby, contain/inside topological relations between cadastral entities.



$a\setminus b$=-∅
a/b=∅
(a) a covers b

$a\setminus b$=∅
a/b=-∅
(b) a covered by b

Figure 1. The effects of difference （\） and difference by（/）

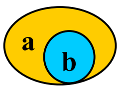
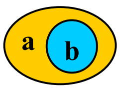
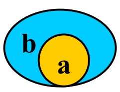
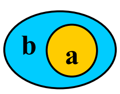
Fc	$a\cup b$ / $a^V\cup b^V$	$a\cap b$ / $a^V\cap b^V$	$a\setminus b$ / $a^V\setminus b^V$	a/b / a^V/b^V	$a\Delta b$ / $a^V\Delta b^V$	Semantic
	-∅	-∅	-∅	∅	-∅	Cover
	-∅	-∅	-∅	∅	-∅	
	-∅	-∅	-∅	∅	-∅	Contain
	-∅	∅	-∅	∅	-∅	
	-∅	-∅	∅	-∅	-∅	Covered by
	-∅	-∅	∅	-∅	-∅	
	-∅	-∅	∅	-∅	-∅	Inside
	-∅	∅	∅	-∅	-∅	

Table 1. Confused topological relations without Voronoi region

2) The effects of symetric difference (Δ)

In voronoi-based spatial algebra, the value"content (f_C)" and "dimension (f_D)"of the symetric difference (Δ)is not sensitive. The effect of symetric difference (Δ)is mainly reflected in the "number of connected components (f_N)" in the relations between line objects. Fig.2 shows such an example, which illustrates that 3 kinds of topological relations can be easily distinguished by the "number of connected components" of their symetric difference (Δ). But like Fig.2 shows, they can also be easily distinguished by difference (/), difference by (\). So symetric difference (Δ) is not necessary in the description of topological relations.

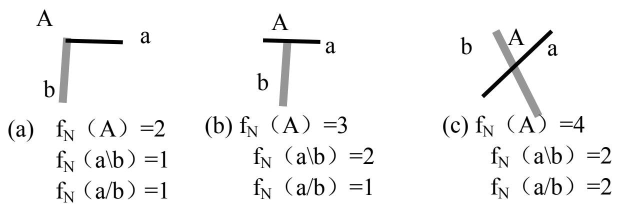


Figure 2. The effect of symetric difference (Δ)

3) The effect of intersection (∩) and union (∪)

In the description of topological relations, intersection is the most important operator, in the early models, such as 4I, 9I, E9I, V9I, intersection is the only operator used to distinguish the topological relations between spatial objects. Intersection almost used in all topological relations' description.

In the description of topological relations, union (∪)can not play any useful role, though it can play an important role in the metric relation, for example, two region with the same size and same shape, the area of the union of them will be descreasing from meet, cover, to equal.

As mentioned above, in the description of topological relations, intersection (∩) is the most important operator, it is used in all objects' topological relations' description; difference (/) or difference by (\) is the second important operator, it plays an important role in the description of cover/ coveredby, contain/inside, etc, topological relations' distinguishing of cadastral entities; symetric difference (Δ)is mainly used in the relations'distinguishing between line objects, and it's effect can be substituted by difference or difference by; union (∪) plays no any useful roles fundamentally.

2.3 Strategies Used in This Study

For the reasons mentioned above, in this stdy, the whole-based approach will be used as a basis. In the set operators, intersection (∩), difference (/) or difference by (\) are essential for all objects, symetric difference (Δ) is mainly used in the line objects, union (∪) needn't to be used.

The models for topological relations have developped from single value (content) to several values (content, dimension and the number of connectd components), from single set operator (intersection) to several operators. Now all set operators have been used. In order to overcome the shortcomings of the whole-based spatial algebra without voronoi region, an additional parameter should be introduced. This parameter must meet the following criteria: 1) like content, dimension and the number of connected components, it must be a topological invariant; 2) it should be sensitive to the result of set operators between spatial objects.

As a result, Euler-number is selected, for it is a topological invariant, and it can reflect the shape of the spatial objects, the holes of region objects, it is the complementarity of content, dimension and number of connected components, as Fig.3 shows.

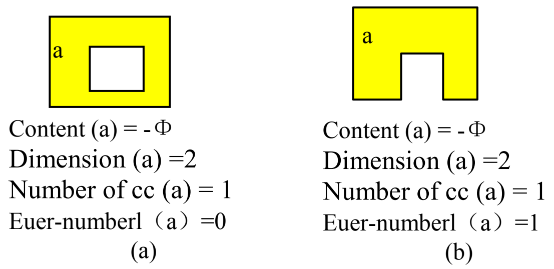


Figure 3. Euler-number

In summary, the basic strategies adopted here are: 1) a spatial object is treated as a whole; 2) Euler-number is introduced to enhance the description capability of the whole-based model; 3) appropriate set operators: intersection, difference or difference by are used to distinguish the topological relations between spatial object at any time, 4) several types of values ars used for the computational result of set operators, e. g. content, dimension, number of connected components and Euler-number.

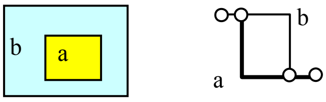
3. A SPATIAL ALGEBRA FOR TOPOLOGICAL RELATIONS BASED ON INTERSECTION AND DIFFERENCE OPERATORS

For the reasons mentioned above, in the set operators, just intersection, difference (/) or difference by (\) are necessary in the description of topological relations. Difference by (\) can be represented by difference (/), for example, A \ B = B / A. Therefore, in the general topological model, only intersection and difference are selected to form a new topological model, which is based on the objects' **W**hole and the set operators of **I**ntersection and **D**ifference.

In existing models, there are three types of values for the result of set operators, i.e. content, dimension, and number of connected components. These values are all used in this model, can be denoted by f_D and f_N. Content and dimension are denoted by f_D, "content" is a quality measure, either "empty (∅)" or "non-empty (-∅)"; "dimension" is a quantitative measure, the values includes 0-dimansional (point), 1-dimensional (line), 2-dimensional (area) and 3-dimensional (solid). For the case of "empty", the dimensional number of (-1) is usually used. So the value range of f_D is {-1, 0, 1, 2, 3}. The number of connected components can be denoted by f_N , which is a quantitative measure at a finer level. In the case of empty, the number is 0, otherwise, the number should be any integer larger than 0. the as showed in EQ (3)

$$R(A,B)=\begin{bmatrix} A\cap B \\ A\setminus B \\ B\setminus A \end{bmatrix} \tag{3}$$

Fig.4 is an example of this model based on content, dimension and number of connected component. (a) a, b are areas, a inside b, the dimension of a∩b is 2 , f_D(a∩b)=2, the number of connected component of a∩b is 1, f_N (a∩b)=1; a\b is empty, f_D (a\b) = -1; the dimension of b\a is 2, f_D (b\a)=2, the number of connected component of b\a is 1, f_N (b\a)=1. (b) a, b are lines, one extremity of a meets at b's middle, and one extremity of b meets at a's middle, the dimension of a∩b is 0 (f_D=0), the number of connected component of a∩b is 2 (f_N=2); the dimension of a\b is 1 (f_D=1), the number of connected component of a\b is 2 (f_N=2); the dimension of b\a is 1 (f_D=1), the number of connected components of b\a is 2 (f_N=2). The values are showed in Fig.4.



$\begin{bmatrix} 21 \\ -1 \\ 21 \end{bmatrix}$ $\begin{bmatrix} 02 \\ 12 \\ 12 \end{bmatrix}$

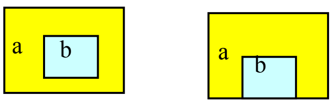
(a) (b)

Figure 4. Examples of this based on the content, dimension, and the number of connected components

4. A SPATIAL ALGEBRA FOR TOPOLOGICAL RELATIONS BASED ON EULER-NUMBER

Howerever, if the values only include content, dimension and number of connected components, there are many relations between spatial objects cann't be distinguished, for example, "include" and "cover", "inside" and "covered by" and so on. Fig.5 shows that "include" and "cover" cann't be distinguished by this model, which is based on content, dimension and number of connected components.

In Fig.5, (a) a include b, (b) a cover b. Though the content, dimension and number of connected components of a∩b, a\b, b\a are equal in Fig.5 (a) and Fig.5 (b), the figure shape of a\b is different, as Fig.6 shows, which can be distinguished by ***Euler number*** easily.



$\begin{bmatrix} 21 \\ 21 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 21 \\ 21 \\ -1 \end{bmatrix}$

(a) (b)

Figure 5. Example: relations cann't be distinguished by WID3 based on content, dimension and number of connected components

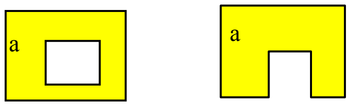
Euler number (sometimes called Euler-Poincare, denoted by *Eul*) is a topological invariant. In basic topology, to n–dimensional simplicial complex L, the Euler number of L, i.e. Eul(L), can be computed by EQ (4):

$$\mathrm{Eul(L)}=\sum_{\mathrm{i}=0}^{\mathrm{n}}(\text{-}1)^{\mathrm{i}}a_i \tag{4}$$

Where a_i is the number of i- dimensional simplicial complex in L. when L is a polyhedren, the ***Euler number*** defines as vertices minus edges plus faces(Armstrong，1983,). To region (L) with holes, ***Euler number*** defines as the number of connected components (C) minus holes (H), as EQ (5) shows:

$$\mathrm{Eul\ (L\ )= C - H} \tag{5}$$

The Euler number of Fig.6 (a) is 0, Fig.6 (b) is 1. Therefore Euler number can help to distinguish the relations cann't be distinguished by , based on content, dimension and number of connected components, as Fig.5 shows.



Eul（a）=0 Eul（a）=1

(a) (b)

Figure 6. Figure shape of a\b

As EQ (5) shows, Euler number can be defined as the number of connected components minus holes. When the holes is 0, Euler number will be equal to the number of connected components. So the number of connected components has been included in Euler number.

Therefore the value of this model based on Euler number can take three different forms, i.e. content, dimension, and Euler number, which also include 2 digits. One denotes the content and dimension of the result, denoted by f_D , the value range is {-1，0，1，2, 3}; The other denotes Euler number, denoted by f_E, when the result of intersection and difference is simple point, line, or region, f_E is equal to the number of connected components. In this study, the value range of f_E is any integer larger than 0, as EQ (6) shows:

$$R(A,B)=\begin{bmatrix} A\cap B:\{\mathrm{f_D},\mathrm{f}_E\} \\ A\setminus B:\{\mathrm{f_D},\mathrm{f}_E\} \\ B\setminus A:\{\mathrm{f_D},\mathrm{f}_E\} \end{bmatrix} \tag{6}$$

5. TOPOLOGICAL RELATIONS BETWEEN SPATIAL OBJECTS WITH THIS MODEL BASED ON EULER NUMBER

In fact, not all the values of EQ (6) are valid in GIS. The spatial objects must satisfy the assumptions as follows: (1) spatial objects are embedded in Euclidean space; (2) a spatial object has only one connected component. There are six kinds of topological relations between spatial objects, i.e. area/ area, line / line, point / point, area / line, area / point, line / point, these relations can be described bythis model based on Euler number.

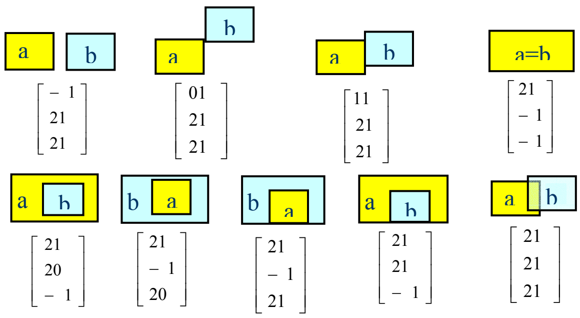


Figure7. Relations between simpleregions

In this study, many relations between spatial objects have been described with the model proposed in this paper, i.e. 9 kinds of simple area / area, as Fig.7 shows; 36 kinds of simple line / line, as Fig.8 shows; 9 kinds of simple area/ simple line, as Fig.9 shows; 2 kinds of point / point, 3 kinds of simple area / point, 3 kinds of point / line, as Fig.10 shows.

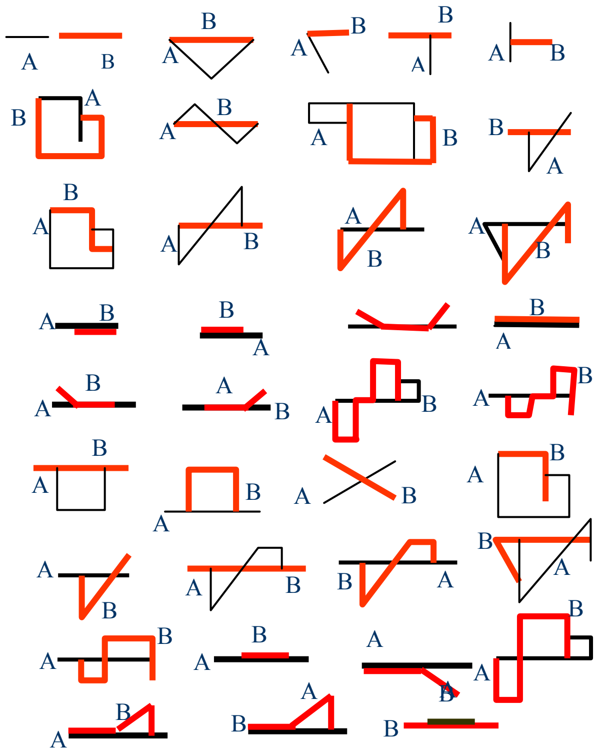


Figure 8. Relations between simple lines

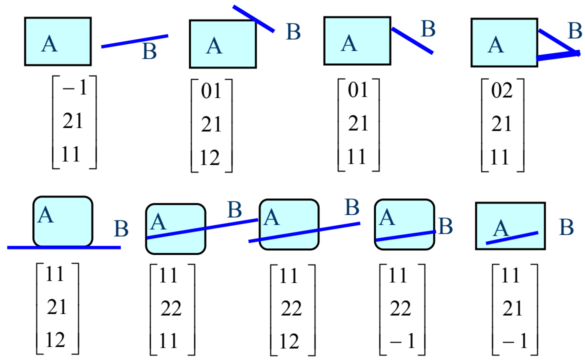


Figure 9. Relations between simple region A and simple line B

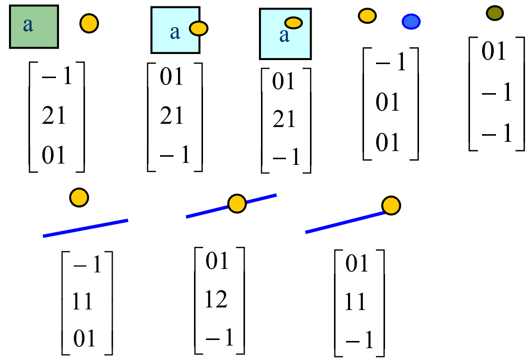


Figure 10. The other relations between simple objects

6. CONCLUSION

In this paper, a topological model for spatial object is proposed, in which: (1) object is treated as a whole; (2) appropriate operators from set operators, i.e. intersection and difference are utilized to distinguish the relations between spatial objects; in order to overcome the deficiency of whole-based algebra without voronoi region, Euler number is introduced. So three types of value are used for the computational results of set operations, i.e. content, dimension, and Euler number.

This approach combines the advantages of existing models.

(1) It just use three expressions to distinguish the the topological relations between spatial objects, is simpler than exsinting models;

(2) By introducing Euler number in the result of difference, overcome the deficiency of whole-based spatial algebra without vornoi region, and Euler number also can be used to descibe the shape of areas, such as simple area (Eul = 1) or area with one hole (Eul = 0), it is more detail than content, dimension and number of connected components;

(3) This model can be used to describe the topological relations of discrete object and associated tessellation object (such as parcel); it also can be used to describe the topological relations of objects at different time, such as paraent-child land parcels, objects before and after changes.

REFERENCES

CHEN Jun, LI Chengming，LI Zhilin and Gold C.M., A Voronoi-based 9-intersection model for spatial relations，*Int. J. GEOGRAPHICAL INFORMATION SCIENCE*，2001,15(3), 201-220.

CHEN Jun, ZHAO Renliang, Spatial Relations in GIS: A Survey on its Key Issues and Research Progress，*ACTA GEODAETICA et CARTOGRAPHICA SINICA*， 1999,28(2),95-102. (In Chinese)

Clementini E. & Di Felice P., A Comparison of Methods for Representing Topological Relationships, *Information Sciences - Applications*,.3(3), 1995, 149-178

Clementini E. et al. Modelling topological spatial relations: Strategies for query processing , *Computers & Graphics*, 18(6),1994, 815-822

Cohn A.G., et al. Exploiting Temporal Continuity in Qyalitive Spatial Calculi,*SPATIAL and TEMPORAL REASONING in GEOGRAPHIC INFORMATION SYSTEMS* , Edited by M. J. EGENHOFER , REGINALD G . GOLIEDGE , New York Oxford , Oxford University Press ,1998

Egenhofer M. and Franzosa R., Point-Set Topological Spatial Relations, *International Journal of Geographical Information Systems*, 1991, 5 (2), 161-174.

Egenhofer, M., A model for detailed binary topological relationships. *Geomatic*, 1993, 47(3)：261-273.

GUO Qing-sheng，Combinatorial Representation of Spatial Relationships on 2D Vector Map, *ACTA GEODAETICA et CARTOGRAPHICA SINICA*, 2000, 29(2), 155-161. (In Chinese)

LI Zhinlin, ZHAO Renliang, CHEN Jun ， An Generic Algebra For Spatial Relations . *Progressing in Natural Science*, 2002, 12(7): 528-536.

SUN Yuguo, The Description and Representation of Topological Relationships with 2D T-String, *Doctor Thesis of Wuhan Technical University of Surveying and Mapping*, 1993. (In Chinese)

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