TOWARDS TOTAL KALMAN FILTERING FOR MOBILE MAPPING

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ABSTRACT:

A dynamic model is the usual modus operandi of a Mobile Mapping System. The model solution, after linearization and discretization, is achieved using the Weighted Least-Squares (WLS) approach, which results in one of the various Kalman filter algorithms. However, implicit in the formulation is that neither the observation equation matrices nor the transition matrices at any epoch contain random entries. As such an assumption cannot always be guaranteed, we here allow random observational errors to enter the respective matrices. We replace the WLS by the Total-Least-Squares (TLS) principle - with or without weights - and apply it to this novel *Dynamic Errors-in-Variables* (DEIV) model, which results in what we call *Total Kalman Filter* (TKF). It promises to offer more representative solutions to the dynamic models of Mobile Mapping Systems over existing versions of Kalman filtering.

1. INTRODUCTION

Mobile Mapping Systems (MMS) are one of the new trends in mapping applications. A MMS has two main components: the mapping sensors and the navigation system. During the survey mission, navigation data (GPS/INS) and mapping data are collected simultaneously, and the navigation data are integrated through *Kalman filtering* followed by backward smoothing using a dynamic model; this is the usual modus operandi of a Mobile Mapping System (MMS). In this process, the model variables of an INS, such as platform position, velocity and attitude, are estimated together with a number of additional random system parameters at a given epoch(s), as well as information about the variance estimates of the navigation solution.

Traditionally, the dynamic model solution, after linearization and discretization, is achieved using the Weighted Least-Squares (WLS) approach, or within the framework of some *Bayesian* considerations, which results in one of the various *Kalman filter* algorithms. In this formulation, the observation equation matrices and the transition matrices at any epoch are not allowed to contain random entries, which cannot always be guaranteed. For non-dynamical solutions, this problem can be ameliorated using a quasi-linear model with *Errors-in-Variables* (*EIV*). Golub and van Loan (1980) showed that the Total Least-Squares Estimate (TLSE) could be obtained by solving an eigenvalue problem for the suitably augmented normal equations matrix from the standard least-squares approach. Several computational methods have been compared recently by Björck et al. (2000). The subject of Total Least-Squares (TLS) in view of EIV-Modeling was also discussed extensively in the books by Golub and van Loan (1996), van Huffel and Vanderwalle (1991), van Huffel (1997), van Huffel and Lemmerling (2002), as well as by Schaffrin and Felus (2005) and Schaffrin (2006) in the context of constraints. However, there is currently no formulation for solving dynamic problems (in real time) within the context of the TLS principle.

In this study, we produce a solution in which we allow random observational errors to be accounted for in the respective matrices of the underlying model. We replace the WLS formulation that leads to the *Discrete Kalman Filter* by the Total-Least-Squares (TLS) principle - with or without weights - and apply it to a new *Dynamic Errors-in-Variables* (*DEIV*) model, whose solution we call *Total Kalman Filtering* (TKF). This new solution, by virtue of considering the errors in variables, offers more representative solutions to the dynamic models of Mobile Mapping Systems over the existing versions of Kalman filters that are currently in use.

In the following sections, we first derive the standard Kalman filter solutions (simple and weighted) using a considerably simpler new approach by making use of *condition equations* and straightforward variance propagation rules familiar to all geodesists. Such an approach provides transparency and simplicity in addition to the insight into the current Kalman filters. The TLS solution within an EIV model is then obtained as result of the traditional Lagrange approach.

Finally, we use a combined approach to derive the TKF as the properly weighted TLS solution to the DEIV model.

2. STANDARD KALMAN FILTER FOR MMS

A typical dynamic model solution, after linearization and discretization, is achieved using the Weighted Least-Squares (WLS) approach that results in one of the various Kalman filter algorithms. In these algorithms, it is *assumed* that the observation equation matrices and the transition matrices at any epoch during the surveys contain non-random entries. The corresponding Dynamic Linear Model underlying such a filter is described in the following subsection.

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2.1 Statistical model of type DLM

At an epoch *i,* a set of *n* observations, denoted by y_i, is *linearly* related to the *m* state variables in x_i (such as position and velocity) through the $n \times m$ *non-stochastic* coefficient matrix A_i and the commensurate observational noise vector e_i :

$$y_i = A_i x_i + e_i \tag{1}$$

The state variables, presumably representing linear dynamics of the system at an epoch i , can be related to the state variables of an earlier epoch *i*-1 through the $m \times m$ state transition matrix $\Phi_{i,i-1}$ as follows:

$$x_i = \Phi_{i-1} x_{i-1} + u_i \tag{2}$$

where we dropped the first index *i* in the stated transition matrix for simplicity. In the above expression, the *state propagation equation*, the *m*-vector u_i denotes the *stochastic* system description noise. It is assumed that additional *stochastic* information, $\tilde{x}_{i-1}$, exists about the state vector x_{i-1} at the initial epoch:

$$\tilde{x}_{i-1} = x_{i-1} + e^0_{i-1} \tag{3}$$

It is further assumed that the random observation errors, the random system errors, and the random errors of the additional information about the state parameters are statistically independent of each other at any epoch as well as over time:

$$\begin{bmatrix} e_i \\ u_i \\ e^0_{i-1} \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_i & 0 & 0 \\ 0 & \Theta_i & 0 \\ 0 & 0 & \Sigma^0_{i-1} \end{bmatrix} \right) \tag{4}$$

The following compact format states the stochastic model underlying the *Standard Kalman Filter*.

$$\boxed{\begin{matrix} y_i = A_i x_i + e_i, \\ x_i = \Phi_{i-1} x_{i-1} + u_i \\ \tilde{x}_{i-1} = x_i + e^0_{i-1}, \end{matrix} \quad , \quad \begin{bmatrix} e_i \\ u_i \\ e^0_{i-1} \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_i & 0 & 0 \\ 0 & \Theta_i & 0 \\ 0 & 0 & \Sigma^0_{i-1} \end{bmatrix} \right)} \tag{5}$$

2.2 Weighted Least-Squares Solution

In the above expression, the known vector of additional information can be multiplied by the state transition matrix from the left and inserted into the state transition equation, eliminating the unknown state parameter vector x_{i-1}. We obtain the following reduced equations by propagating the corresponding dispersion (variance/covariance) matrices:

$$\boxed{\begin{aligned} & y_i = A_i x_i + e_i, \\ & \breve{x}_i := \Phi_{i-1}\tilde{x}_{i-1} = x_i - (u_i - \Phi_{i-1} e^0_{i-1}), \\ & \begin{bmatrix} e_i \\ u_i - \Phi_{i-1} e^0_{i-1} \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_i & 0 \\ 0 & \Theta_i + \Phi_{i-1}\Sigma^0_{i-1}\Phi^T_{i-1} \end{bmatrix} \right) \end{aligned}} \tag{6}$$

Further elimination of the unknown state parameter vector x_i from the above expression results in:

$$\boxed{\begin{aligned} & e_i + A_i(u_i - \Phi_{i-1} e^0_{i-1}) - (y_i - A_i \breve{x}_i) = 0, \\ & e_i + A_i(u_i - \Phi_{i-1} e^0_{i-1}) \sim \left(0,\, \Sigma_i + A_i(\Theta_i + \Phi_{i-1}\Sigma^0_{i-1}\Phi^T_{i-1})A^T_i\right) \end{aligned}} \tag{7}$$

We observe that this last expression is a set of condition equations among all the unknown stochastic error parameters. Their least-squares solution gives:

$$\begin{aligned} \begin{bmatrix} \tilde{e}_i \\ \tilde{u}_i \\ \tilde{e}^0_{i-1} \end{bmatrix} = \begin{bmatrix} \Sigma_i & 0 & 0 \\ 0 & \Theta_i & 0 \\ 0 & 0 & \Sigma^0_{i-1} \end{bmatrix} \begin{bmatrix} I_i \\ A^T_i \\ -\Phi^T_{i-1} A^T_i \end{bmatrix} \cdot \\ \cdot [\Sigma_i + A_i(\Theta_i + \Phi_{i-1}\Sigma^0_{i-1}\Phi^T_{i-1})A^T_i]^{-1}(y_i - A_i \breve{x}_i) \end{aligned} \tag{8}$$

from which the adjusted state parameters are computed as follows:

$$\tilde{x}_i = \breve{x}_i + (\tilde{u}_i - \Phi_{i-1}\tilde{e}^0_{i-1}) = \breve{x}_i + \mathrm{K}_i z_i \tag{9}$$

where

$$\begin{aligned} \mathrm{K}_i &:= (\Theta_i + \Phi_{i-1}\Sigma^0_{i-1}\Phi^T_{i-1})A^T_i \cdot \\ &\quad \cdot [\Sigma_i + A_i(\Theta_i + \Phi_{i-1}\Sigma^0_{i-1}\Phi^T_{i-1})A^T_i]^{-1}, \\ z_i &:= (y_i - A_i \breve{x}_i). \end{aligned} \tag{10}$$

Note that

$$E\{\tilde{x}_i\} = E\{\breve{x}_i\} = E\{x_i\}. \tag{11}$$

Hence, the mean squared error (MSE) matrix of the estimated state vector is equal to its error dispersion matrix and is given by:

$$\begin{aligned} MSE\{\tilde{x}_i\} &= D\{\tilde{x}_i - x_i\} = D\{(\tilde{x}_i - \breve{x}_i) - (x_i - \breve{x}_i)\} = \\ &= D\{K_i z_i - (u_i - \Phi_{i-1} e^0_{i-1})\} = \\ &= D\{K_i e_i + (K_i A_i - I_m)(u_i - \Phi_{i-1} e^0_{i-1})\} = \\ &= K_i \Sigma_i K^T_i + (I_m - K_i A_i)(\Theta_i + \Phi_{i-1}\Sigma^0_{i-1}\Theta^T_{i-1})(I_m - K_i A_i)^T \end{aligned} \tag{12}$$

It reduces, after some algebraic manipulations, to:

$$\boxed{D\{\tilde{x}_i - x_i\} = (I_m - K_i A_i)(\Theta_i + \Phi_{i-1}\Sigma^0_{i-1}\Phi^T_{i-1}) =: \Sigma^0_i} \tag{13}$$

3. TOTAL LEAST-SQUARES IN THE ERRORS-IN-VARIABLES MODEL

We will use the following synoptic solution to the *Total Least-Squares* problem in an *Errors-in-Variables Model*, TLS in EIV, which will later lead to the newly proposed Total Kalman Filtering.

3.1 Statistical Model of Type EIV

The corresponding observation equations are stated as:

$$\boxed{y = (A - E_A)_{n\times m}\,\xi + e, \quad n > m = rank(A),} \qquad (14)$$

with the following assumed random error properties for the observational noise:

$$\boxed{\begin{bmatrix} e \\ e_A := vecE_A \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_0^2 \underset{n\times n}{Q} & 0 \\ 0 & \sigma_0^2 I_m \otimes \underset{n\times n}{Q} \end{bmatrix} \right).} \qquad (15)$$

As before, y denotes the $n \times 1$ vector of observations that are linearly related to the $m \times 1$ vector of unknown parameters, ξ, through the $n \times m$ stochastic (observed) coefficient matrix A of rank m, and e and e_A are the corresponding observational noise vectors with an unknown variance factor σ_0^2. Rearranging (14) gives the non-linear *Gauss-Helmert* model (Helmert, 1907):

$$y = A\xi + \begin{bmatrix} I_n & -(\xi^T \otimes I_n) \end{bmatrix} \begin{bmatrix} e \\ e_A \end{bmatrix}. \qquad (16)$$

We now form the following *Lagrange target function* associated with the Total Least-Squares (TLS) principle, with $P := Q^{-1}$ and λ as the $n \times 1$ vector of *Lagrange* multipliers:

$$\begin{aligned} \Phi(e, e_A, \xi, \lambda) &:= e^T P e + e_A^T (I \otimes P) e_A + \\ &+ 2\lambda^T [y - e - (\xi^T \otimes I_m)(vecA - e_A)] = stationary \end{aligned} \qquad (17)$$

The following partial derivatives of the Lagrange target function establish the stationarity conditions.

Euler-Lagrange Conditions:

$$\frac{1}{2}\frac{\partial \Phi}{\partial e} = P\tilde{e} - \hat{\lambda} \doteq 0 \quad \Rightarrow \boxed{\tilde{e} = Q\hat{\lambda}} \qquad (18)$$

$$\begin{aligned} &\frac{1}{2}\frac{\partial \Phi}{\partial e_A} = (I_m \otimes P)\tilde{e}_A + (\hat{\xi} \otimes I_n)\hat{\lambda} \doteq 0 \\ &\Rightarrow \tilde{e}_A = -(\hat{\xi} \otimes Q)\hat{\lambda} \quad \Rightarrow \boxed{\tilde{E}_A = -Q\hat{\lambda}\hat{\xi}^T} \end{aligned} \qquad (19)$$

$$\begin{aligned} &\frac{1}{2}\frac{\partial \Phi}{\partial \xi} = -(A - \tilde{E}_A)^T \hat{\lambda} \doteq 0 \\ &\Rightarrow A^T \hat{\lambda} = \tilde{E}_A^T \hat{\lambda} = -\hat{\xi}(\hat{\lambda}^T Q \hat{\lambda}) \end{aligned} \qquad (20)$$

$$\begin{aligned} &\frac{1}{2}\frac{\partial \Phi}{\partial \lambda} = y - \tilde{e} - A\hat{\xi} + \tilde{E}_A\hat{\xi} \doteq 0 \\ &\Rightarrow \boxed{P(y - A\hat{\xi})(1 + \hat{\xi}^T \hat{\xi})^{-1} = \hat{\lambda}} \end{aligned} \qquad (21)$$

$$\begin{aligned} &\Rightarrow A^T P(y - A\hat{\xi}) = A^T P(\tilde{e} - \tilde{E}_A\hat{\xi}) = A^T \hat{\lambda} \cdot (1 + \hat{\xi}^T \hat{\xi}) \\ &\Rightarrow A^T \hat{\lambda} = A^T P(y - A\hat{\xi})(1 + \hat{\xi}^T \hat{\xi})^{-1} \end{aligned}$$

Using (20) and (21) we obtain:

$$\begin{aligned} &A^T P(y - A\hat{\xi}) = -\hat{\xi}(\hat{\lambda}^T Q \hat{\lambda})(1 + \hat{\xi}^T \hat{\xi}) = \\ &= -\hat{\xi} \cdot (1 + \hat{\xi}^T \hat{\xi})^{-1} (y - A\hat{\xi})^T P (y - A\hat{\xi}) = -\hat{\xi} \cdot \hat{\nu} \end{aligned} \qquad (22)$$

where

$$\boxed{\hat{\nu} := (\hat{\lambda}^T Q \hat{\lambda})(1 + \hat{\xi}^T \hat{\xi}) = \tilde{e}^T P \tilde{e} + \tilde{e}_A^T (I_m \otimes P)\tilde{e}_A = TSSR} \qquad (23)$$

(*TSSR*: Total Sum of Squared Residuals). Denoting

$$\boxed{c := A^T P y = (N - \hat{\nu} I_m) \cdot \hat{\xi} \quad \text{for } N := A^T P A,} \qquad (24)$$

the first-order approximate solution (as a result of neglecting the randomness of $\hat{\nu}$) of the dispersion matrix is given by

$$\boxed{\begin{aligned} D\{\hat{\xi}\} &\approx \sigma_0^2 (N - \hat{\nu} I_m)^{-1} N (N - \hat{\nu} I_m)^{-1} = \\ &= \sigma_0^2 (N - \hat{\nu} I_m)^{-1} + \sigma_0^2 \hat{\nu} (N - \hat{\nu} I_m)^{-2}. \end{aligned}} \qquad (25)$$

4. TOTAL KALMAN FILTER

The extension of the previously discussed TLS approach to a DEIV model is now in order.

4.1 DEIV - Observation Equations

We consider the following observation equations at any epoch i, together with the state transition equations. Again, the coefficient matrix in front of the unknown state vector is stochastic in nature (but not the state transition matrix itself):

$$\boxed{\begin{aligned} &y_i = (A_i - E_{A_i}) x_i + e_i, \\ &\breve{x}_i := \Phi_{i-1} \tilde{x}_{i-1} = x_i - (u_i - \Phi_{i-1} e_{i-1}^0), \end{aligned}} \qquad (26)$$

and

$$\boxed{\begin{bmatrix} e_i \\ vecE_{A_i} = e_{A_i} \\ u_i - \Phi_{i-1}e_{i-1}^0 \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_i & 0 & 0 \\ 0 & I_m \otimes \Sigma_i & 0 \\ 0 & 0 & \Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T \end{bmatrix} \right).} \tag{27}$$

The corresponding *Lagrange target function*, following Schaffrin (1995), is now given for $P_i := \Sigma_i^{-1}$ by:

$$\begin{aligned} &\Phi(e_i, e_{A_i}, x_i, \lambda_i, \mu_i) := e_i^T P_i e_i + e_{A_i}^T (I_m \otimes P_i) e_{A_i} + \\ &+ 2\lambda_i^T [y_i - e_i - A_i x_i + (x_i^T \otimes I_n) e_{A_i}] - \\ &- 2\mu_i^T (\breve{x}_i - x_i) - \mu_i^T (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0 \Phi_{i-1}^T)\mu_i = stationary \end{aligned} \tag{28}$$

where the last term is due to the *stochastic constraints* in (26). This target function is used to derive the following *Euler-Lagrange necessary conditions*.

Euler-Lagrange Necessary Conditions:

$$\frac{1}{2}\frac{\partial \Phi}{\partial e_i} = P_i \tilde{e}_i - \hat{\lambda}_i \doteq 0 \quad \Rightarrow \tilde{e}_i = \Sigma_i \hat{\lambda}_i \tag{29}$$

$$\begin{aligned} &\frac{1}{2}\frac{\partial \Phi}{\partial e_{A_i}} = (I_m \otimes P_i)\tilde{e}_{A_i} + (\tilde{x}_i \otimes I_n)\hat{\lambda}_i \doteq 0 \\ &\Rightarrow \tilde{e}_{A_i} = -(\tilde{x}_1 \otimes \Sigma_i)\hat{\lambda}_i \\ &\Rightarrow \tilde{E}_{A_i} = -\Sigma_i \hat{\lambda}_i \tilde{x}_i^T = -\tilde{e}_i \tilde{x}_i^T \end{aligned} \tag{30}$$

$$\begin{aligned} &\frac{1}{2}\frac{\partial \Phi}{\partial x_i} = -(A_i - \tilde{E}_{A_i})^T \hat{\lambda}_i + \hat{\mu}_i \doteq 0 \\ &\quad \Rightarrow \hat{\mu}_i - A_i^T \hat{\lambda}_i = -\tilde{E}_{A_i}^T \hat{\lambda}_i = \tilde{x}_i (\hat{\lambda}_i^T \Sigma_i \hat{\lambda}_i) \end{aligned} \tag{31}$$

$$\begin{aligned} &\frac{1}{2}\frac{\partial \Phi}{\partial \lambda_i} = y_i - A_i \tilde{x}_i - \tilde{e}_i + (\tilde{x}_i^T \otimes I_n)\tilde{e}_{A_i} \doteq 0 \\ &\qquad \Rightarrow P_i(y_i - A_i \tilde{x}_i) = P_i(\tilde{e}_i - \tilde{E}_{A_i}\tilde{x}_i) = \hat{\lambda}_i (1 + \tilde{x}_i^T \tilde{x}_i) \\ &\qquad \Rightarrow \boxed{\hat{\lambda}_i = P_i (y_i - A_i \tilde{x}_i)(1 + \tilde{x}_i^T \tilde{x}_i)^{-1}} \end{aligned} \tag{32}$$

$$\begin{aligned} &\frac{1}{2}\frac{\partial \Phi}{\partial \mu_i} = -\breve{x}_i + \tilde{x}_i - (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0 \Phi_{i-1}^T)\hat{\mu}_i \doteq 0 \\ &\Rightarrow \boxed{\tilde{x}_i = \breve{x}_i + (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0 \Phi_{i-1}^T)\hat{\mu}_i} \end{aligned} \tag{33}$$

Using (31) and (32) we obtain:

$$\boxed{\begin{aligned} &A_i^T P_i (y_i - A_i \tilde{x}_i) - \hat{\mu}_i (1 + \tilde{x}_i^T \tilde{x}_i) = (A_i^T \hat{\lambda}_i - \hat{\mu}_i)(1 + \tilde{x}_i^T \tilde{x}_i) = \\ &= -\tilde{x}_i (1 + \tilde{x}_i^T \tilde{x}_i)^{-1} (y_i - A_i \tilde{x}_i)^T P_i (y_i - A_i \tilde{x}_i) =: -\tilde{x}_i \cdot \hat{v}_i \end{aligned}} \tag{34}$$

where the coefficient $\hat{v}_i$ is defined by:

$$\begin{aligned} &\hat{v}_i := [y_i^T P_i (y_i - A_i \tilde{x}_i) - \tilde{x}_i^T A_i^T P_i (y_i - A_i \tilde{x}_i)] \cdot (1 + \tilde{x}_i^T \tilde{x}_i)^{-1} = \\ &= [y_i^T P_i (y_i - A_i \tilde{x}_i) + (\tilde{x}_i^T \tilde{x}_i)\hat{v}_i] \cdot (1 + \tilde{x}_i^T \tilde{x}_i)^{-1} - \tilde{x}_i^T \hat{\mu}_i , \end{aligned} \tag{35}$$

thus leading to :

$$\boxed{\begin{aligned} &\hat{v}_i := y_i^T P_i (y_i - A_i \tilde{x}_i) - \tilde{x}_i^T \hat{\mu}_i (1 + \tilde{x}_i^T \tilde{x}_i)^{-1} = [y_i^T P_i (y_i - A_i \tilde{x}_i) - \\ &\quad - \tilde{x}_i^T (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0 \Phi_{i-1}^T)^{-1} (\tilde{x}_i - \breve{x}_i)(1 + \tilde{x}_i^T \tilde{x}_i) \end{aligned}} \tag{36}$$

after taking (33) into account which generalizes the usual Kalman filer formula (9). Obviously, (34) and (36) have to be solved iteratively, and a formula (approximate) for the mean squared error matrix $\Sigma_i^0 = MSPE(\tilde{x}_i)$ still needs to be developed. Instead, let us follow a different route that should provide the same results in another form.

4.2 Combining the EIV-Observations with the State Equations

Here our model reads:

$$\boxed{\begin{aligned} y_i - A_i \breve{x}_i = &-E_{A_i} . [\breve{x}_i + (u_i - \Phi_{i-1}e_{i-1}^0)] + \\ &+ [e_i + A_i (u_i - \Phi_{i-1}e_{i-1}^0)] \end{aligned}} \tag{37}$$

with

$$\boxed{\begin{bmatrix} e_i \\ e_{A_i} = vecE_{A_i} \\ u_i - \Phi_{i-1}e_{i-1}^0 \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_i & 0 & 0 \\ 0 & I_m \otimes \Sigma_i & 0 \\ 0 & 0 & \Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T \end{bmatrix} \right)} \tag{38}$$

We consider the following Lagrange target function with $P_i := \Sigma_i^{-1}$:

$$\begin{aligned} &\Phi(e_i, e_{A_i}, u_i - \Phi_{i-1}e_{i-1}^0, \lambda_i) := e_i^T P_i e_i + e_{A_i}^T (I_m \otimes P_i) e_{A_i} + \\ &\quad + (u_i - \Phi_{i-1}e_{i-1}^0)^T (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0 \Phi_{i-1}^T)^{-1} (u_i - \Phi_{i-1}e_{i-1}^0) + \\ &\quad + 2\lambda_i^T [(y_i - A_i \breve{x}_i) - e_i - A_i (u_i - \Phi_{i-1}e_{i-1}^0) + \\ &\quad + E_{A_i}(u_i - \Phi_{i-1}e_{i-1}^0) + E_{A_i}\breve{x}_i] = stationary \end{aligned} \tag{39}$$

which implies the following *Euler-Lagrange necessary conditions* for the stationarity solution.

Euler-Lagrange Necessary Conditions:

$$\frac{1}{2}\frac{\partial \Phi}{\partial e_i} = P_i \tilde{e}_i - \hat{\lambda}_i \doteq 0 \quad \Rightarrow \tilde{e}_i = \Sigma_i \hat{\lambda}_i \tag{40}$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial e_{A_i}} = (I_m \otimes P_i)\tilde{e}_{A_i} + \left[\left(\breve{x}_i + \tilde{u}_i - \Phi_{i-1}\tilde{e}_{i-1}^0\right) \otimes I_n\right]\hat{\lambda}_i \doteq 0$$

$$\Rightarrow \tilde{e}_{A_i} = -\left[\left(\breve{x}_i + \tilde{u}_i - \Phi_{i-1}\tilde{e}_{i-1}^0\right) \otimes \Sigma_i\right] \cdot \hat{\lambda}_i$$

$$\Rightarrow \tilde{E}_{A_i} = -\Sigma_i \hat{\lambda}_i \left(\breve{x}_i + \tilde{u}_i - \Phi_{i-1}\tilde{e}_{i-1}^0\right)^T = -\tilde{e}_i\ \tilde{x}_i^T \qquad (41)$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial (u_i - \Phi_{i-1}e_{i-1}^0)} = (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T)^{-1}(\tilde{u}_i - \Phi_{i-1}\tilde{e}_{i-1}^0) -$$

$$-(A_i - \tilde{E}_{A_i})^T \cdot \hat{\lambda}_i \doteq 0$$

$$\Rightarrow \boxed{\begin{aligned}&\tilde{u}_i - \Phi_{i-1}\tilde{e}_{i-1}^0 = (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T) \cdot \\ &\cdot [A_i^T \hat{\lambda}_i + \tilde{x}_i(\hat{\lambda}_i^T \Sigma_i \hat{\lambda}_i)] = \tilde{x}_i - \breve{x}_i\end{aligned}} \qquad (42)$$

$$\frac{1}{2}\frac{\partial \Phi}{\partial \lambda_i} = y_i - A_i\breve{x}_i - \tilde{e}_i - A_i(\tilde{u}_i - \Phi_{i-1}\tilde{e}_{i-1}^0) + \tilde{E}_{A_i}\tilde{x}_i \doteq 0$$

$$\Rightarrow P_i(y_i - A_i\breve{x}_i) = \hat{\lambda}_i + P_iA_i\,(\tilde{x}_i - \breve{x}_i) + \hat{\lambda}_i(\tilde{x}_i^T\tilde{x}_i) =$$

$$= \hat{\lambda}_i \cdot (1 + \tilde{x}_i^T\tilde{x}_i) + P_iA_i\left(\tilde{x}_i - \breve{x}_i\right) \qquad (43)$$

$$\Rightarrow \boxed{\hat{\lambda}_i = P_i(y_i - A_i\tilde{x}_i) \cdot (1 + \tilde{x}_i^T\tilde{x}_i)^{-1}} \qquad (44)$$

and thus:

$$\tilde{e}_i = (y_i - A_i\tilde{x}_i) \cdot (1 + \tilde{x}_i^T\tilde{x}_i)^{-1}, \qquad (45)$$

$$\tilde{E}_{A_i} = -(y_i - A_i\tilde{x}_i)(1 + \tilde{x}_i^T\tilde{x}_i)^{-1} \cdot \tilde{x}_i^T = -\tilde{e}_i \cdot \tilde{x}_i^T. \qquad (46)$$

From (42) and (44) we obtain:

$$\boxed{\begin{aligned}&\tilde{x}_i - \breve{x}_i = (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T)\left[A_i^T\hat{\lambda}_i + \tilde{x}_i \cdot (\hat{\lambda}_i^T\Sigma_i\hat{\lambda}_i)\right] = \\ &= (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T)[A_i^TP_i(y_i - A_i\tilde{x}_i) + \tilde{x}_i \cdot \hat{\nu}_i] \cdot (1 + \tilde{x}_i^T\tilde{x}_i)^{-1}\end{aligned}} \qquad (47)$$

where

$$\boxed{\begin{aligned}&\hat{\nu}_i := (\lambda_i^T\Sigma_i\hat{\lambda}_i)(1 + \tilde{x}_i^T\tilde{x}_i) = \\ &= (1 + \tilde{x}_i^T\tilde{x}_i)^{-1}(y_i - A_i\tilde{x}_i)^T P_i(y_i - A_i\tilde{x}_i)\end{aligned}} \qquad (48)$$

so that

$$\begin{aligned}A_i^TP_i(y_i - A_i\breve{x}_i) &= A_i^TP_iA_i(\tilde{x}_i - \breve{x}_i) - \tilde{x}_i \cdot \hat{\nu}_i + \\ &+ (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T)^{-1}(\tilde{x}_i - \breve{x}_i)(1 + \tilde{x}_i^T\tilde{x}_i) = \\ &= [A_i^TP_iA_i + (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T)^{-1}](\tilde{x}_i - \breve{x}_i) - \\ &- \tilde{x}_i \cdot \hat{\nu}_i + (\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T)^{-1}(\tilde{x}_i - \breve{x}_i)(\tilde{x}_i^T\tilde{x}_i)\end{aligned} \qquad (49)$$

from which we conclude:

$$\boxed{\begin{aligned}&\tilde{x}_i = \breve{x}_i + \mathrm{K}_i z_i + (I_m - \mathrm{K}_iA_i) \cdot \\ &\cdot \left[(\Theta_i + \Phi_{i-1}\Sigma_{i-1}^0\Phi_{i-1}^T) \cdot \tilde{x}_i\hat{\nu}_i - (\tilde{x}_i - \breve{x}_i)(\tilde{x}_i^T\tilde{x}_i)\right]\end{aligned}} \qquad (50)$$

with K_i and z_i being defined as in (10).

We observe that, in this expression, the last term is the modification due to the TLS principle.

5. CONCLUSION

The solution for the MSE matrix of TKF is complex but manageable. An approximate solution for the MSE matrix is available and under testing. Alternatively, numerical estimates for the MSE matrix can always be computed by bootstrapping.

In this study, we derived a detailed TLS solution to the DEIV model, and called it *Total Kalman Filter* (*TKF*). It is evident that the TKF solution is an improvement over the existing Kalman filters by virtue of inclusion of the errors in variables in the solution. Nevertheless, additional quantitative studies are needed to quantify its advantages for solutions under varying observations and - at a later stage - state noise for a variety of dynamic models.

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