

3D ASPECTS OF 2D EPIPOLAR GEOMETRY

Ilias Kalisperakis, George Karras, Lazaros Grammatikopoulos

Laboratory of Photogrammetry, Department of Surveying,
National Technical University of Athens (NTUA), GR-15780 Athens, Greece
E-mail: ilias_k@central.ntua.gr, gkarras@central.ntua.gr, lazaros@central.ntua.gr

KEY WORDS: calibration, relative orientation, epipolar geometry

ABSTRACT

Relative orientation in a stereo pair (establishing 3D epipolar geometry) is generally described as a rigid body transformation, with one arbitrary translation component, between two formed bundles of rays. In the uncalibrated case, however, only the 2D projective pencils of epipolar lines can be established from simple image point homologies. These may be related to each other in infinite variations of perspective positions in space, each defining different camera geometries and relative orientation of image bundles. It is of interest in photogrammetry to also approach the 3D image configurations embedded in 2D epipolar geometry in a Euclidean (rather than a projective-algebraic) framework. This contribution attempts such an approach initially in 2D to propose a parameterization of epipolar geometry; when fixing some of the parameters, the remaining ones correspond to a ‘circular locus’ for the second epipole. Every point on this circle is related to a specific direction on the plane representing the intersection line of image planes. Each of these points defines, in turn, a circle as locus of the epipole in space (to accommodate all possible angles of intersection of the image planes). It is further seen that knowledge of the lines joining the epipoles with the respective principal points suffices for establishing the relative position of image planes and the direction of the base line in model space; knowledge of the actual position of the principal points allows full relative orientation and camera calibration of central perspective cameras. Issues of critical configuration are also addressed. Possible future tasks include study of different a priori knowledge as well as the case of the image triplet.

1. INTRODUCTION

In photogrammetric textbooks a typical definition of the task of relative orientation (RO) is that of establishing the relative position of two – already formed – homologue bundles of rays (involving 5 independent parameters). The object may then be reconstructed by bundle intersection in an arbitrarily oriented and scaled model space. In this sense, certain explicit or implicit assumptions are made:

- In order to establish RO, the camera interior orientation (IO) must be fully known beforehand. Knowledge of IO is also a prerequisite for linear algorithms for estimating RO – in fact equivalent to the computation of the ‘essential matrix’, as it came to be known in computer vision literature – which have been presented in photogrammetry (Thompson, 1959; Stefanovic, 1973; Khlebnikova, 1983).

- Determination of epipolar lines presupposes knowledge of both IO and RO. For instance: “If relative orientation is known for a given stereo pair, the coplanarity condition can be used to define epipolar lines” (Wolf & DeWitt, 2000). Or: “The epipolar lines can be determined after the photographs have been relatively oriented” (Mikhail et al., 2001).

Thus, most photogrammetric textbooks (rather understandably, in the context of routine mapping tasks using metric cameras) restrict the definition of RO to that for calibrated images. Currently, however, a more general view on RO is also adopted. For instance, according to the new edition of the *Manual of Photogrammetry* (McGlone, 2004) “the relative orientation of two uncalibrated straight-line-preserving cameras is characterized by 7 independent parameters. An object can be reconstructed only up to a spatial homography”. It is also noted there that, 150 years ago, M. Chasles had detected the 1D homography between corresponding pencils of epipolar lines, whose 3 parameters combine with the 4 parameters defining the epipoles to yield the total of 7 independent parameters required for establishing RO in the uncalibrated case. It is further stated that 7 pairs of homologue points allow finding RO in uncalibrated stereopairs. It needs to be noted that ‘relative orientation’ stands here for something

more general than the conventional photogrammetric concept (since no unique spatial relationship between the two images is fixed); it actually means ‘recovery of 2D epipolar geometry’.

Clearly, it is thanks to extensive research in the field of computer vision that this point of view is being (re)introduced into the photogrammetric literature. In particular, Faugeras (1992) and Hartley (1992) have demonstrated that the 2D epipolar geometry of an image pair may still be established even with unknown IO. The ‘fundamental matrix’ F – having 7 independent parameters, found from simple point homologies – establishes the epipolar constraint on the uncalibrated pair and allows projectively distorted, i.e. non Euclidean, 3D reconstructions (Hartley & Zisserman, 2000, Faugeras & Luong, 2001).

Undoubtedly, the notion of the fundamental matrix has allowed a deeper insight into the structure of the stereopair. In fact – although somehow obscured in the many decades of technological advance and massive photogrammetric production – this generalization of the term ‘relative orientation’ to include the uncalibrated case (and thus signify the establishment of 2D epipolar geometry) is not unfamiliar to photogrammetry. Thus, in the framework of projective geometry Bender (1971) had formulated the equivalent of the fundamental matrix, which represents “the most general relative orientation of two photos”. He also explained that use of one arbitrary camera matrix leads to construction of a model space related to the real object via a 3D projectivity. He concluded that its 15 parameters, added to the 7 of relative orientation, yield the 22 parameters (two DLT matrices), which fully relate two individual images to object space. Yet, it was Sebastian Finsterwalder who – in one of his remarkable publications – had already shown that, given the two epipoles, one can reconstruct an ‘auxiliary’ 3D object which is collinear to that depicted on the images, since all straight lines of the real object also correspond to straight lines on the two perspective projections from which this ‘auxiliary’ object is reconstructed. He pointed out that, assuming central perspective cameras, ⁵ such projective reconstructions are possible from an uncalibrated image pair (Finsterwalder, 1899).

Notwithstanding the elegance and ‘compactness’ of essentially algebraic approaches, it is believed that the more geometric reasoning of Finsterwalder (also adopted in Rinner & Burkhardt, 1972) is indeed also useful to photogrammetry. It might help further illuminate the actual 3D geometry of the stereo pair, by indicating the countless combinations of relative – in its conventional meaning – and interior orientations embedded in one and the same 2D epipolar geometry. Besides, it could also further clarify why partial knowledge of interior orientation can allow recovering camera geometry from simple image point correspondences. It is from this point of view that the authors wish to address here certain aspects of two-image geometry.

2. THE GENERAL CASE

The base line o , defined by the projection centres O_1 and O_2 of a stereopair, intersects the two image planes ε_1 , ε_2 in the respective epipoles e_1 and e_2 (Fig. 1). Epipolar planes, defined by the base and each imaged object point (such as P), intersect the image planes in homologue epipolar lines passing through the epipoles. Corresponding epipolar lines intersect on the intersection line (g) of the two image planes; the pencils of epipolar rays in both images are, therefore, projective (Hallert, 1960).

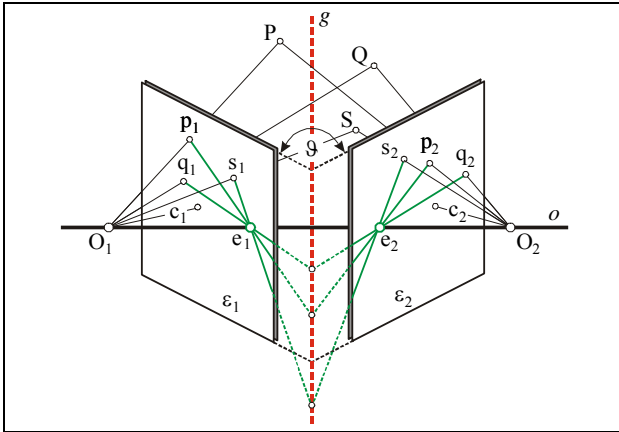


Figure 1. Epipolar geometry.

When only a sufficient number of image point correspondences are at hand, determination of the fundamental matrix allows establishing the epipoles (and consequently the projective pencils of epipolar lines). Once these two pencils are brought to some position in which they are perspective to each other, it is seen in Fig. 1 that changes of angle ϑ between the two image planes do not affect the perspective position of the pencils (they still intersect on g) and the coincidence of the epipolar planes. Thus, one may for the moment address the problem in 2D (i.e. set $\vartheta = \pi$).

First, it is assumed that the two image planes are not parallel (g is not at infinity) and that no image is parallel to the base line o (no epipole is at infinity). Thus, referring to Fig. 2, the pencil of e_1 may be intersected with any line g on image plane ε_1 not passing through e_1 . No generality is lost if only lines through some fixed point K are considered, since the position of K only affects scale. For convenience, K is fixed on some epipolar ray through e_1 and all lines g are characterised by the angle δ they form with this ray. A line $g(\delta)$ intersects two other epipolar rays of e_1 in points A and B . Epipole e_2 can be constructed as the intersection of the two circular arcs c_a and c_b which see segments KA and KB under the respective angles α_2 and $\alpha_2 + \beta_2$ of the epipolar pencil of the second image. All other couples of homologue epipolar rays also intersect on g (cross ratio constraint).

It is to note that a valid second location also exists for e_2 , on the

other side of line g , as the intersection of the circular arcs c_a and c_b if these are mirrored with respect to g . However, this could be disregarded since it corresponds to $\vartheta = 0$. A further remark is that, for some direction δ of line g , any translation of K (point of rotation of g) simply slides the position of e_2 along e_1e_2 (i.e. direction δ defines a line through e_1 as the locus of e_2).

Following Fig. 2, it can be also shown that the geometric locus for epipole e_2 is a circle c_k (which also contains e_1). Consider, for instance, the intersections M and N of epipolar lines e_1A and e_1B with c_a and c_b , respectively. In c_a point A views chord KM under the angle $\alpha_1 + \delta$, while e_2 sees it under the supplementary angle. In c_b point B views chord KN under the angle $\alpha_1 + \beta_1 + \delta$, while e_2 sees it under the supplementary angle. Thus, the angular difference $\text{Ke}_2\text{M} - \text{Ke}_2\text{N} = \text{KBN} - \text{KAM}$, namely segment MN is seen from e_2 under the fixed angle β_1 (or, if e_2 lies between M and N , under angle $\pi - \beta_1$).

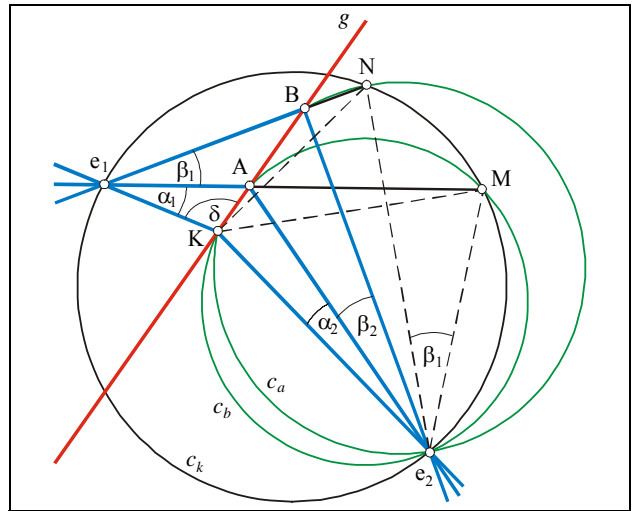


Figure 2. Epipolar geometry on the plane.

Thus, quite independently from the direction δ of line g , every point K fixes a circle c_k as the geometric locus of the epipole e_2 . This circle can be constructed from e_1 and the two points M , N which are fixed by angles α_2 and β_2 of the pencil through e_2 . Indeed, as seen from Fig. 2, M views segment KA under angle α_2 regardless of δ , i.e. of the actual position of A on epipolar line e_1A ; the same is true for point N , which sees segment KB under angle $\alpha_2 + \beta_2$ regardless of the position of B on epipolar line e_1B . Fixing K also fixes these two points (and all other similarly defined points corresponding to other epipolar lines of the pencil through e_1) and, as a consequence, c_k can be constructed.

Thus, the 7 degrees of freedom in 2D epipolar geometry (fundamental matrix) may be parameterized in the following way. The epipoles on ε_1 and ε_2 are fixed with 4 parameters. The remaining 3 degrees of freedom on the plane, which bring the two pencils of epipolar rays in perspective position (they intersect on a line g of direction δ rather than on a conic section), may be geometrically described as follows. For a fixed point K – its location only affects scale and is irrelevant to relative orientation – 2 parameters define with epipole e_1 the circle c_k , whose points are valid locations of epipole e_2 . The third parameter is the rotation which, for any valid e_2 , brings the corresponding epipolar ray to pass through K . This constrains the intersection of the two projective pencils on a line $g(\delta)$. In this sense, and disregarding scale, all possible positions of epipole e_2 can be grasped as a circular movement of all points of c_k around their corresponding line $g(\delta)$ and normal to it. This includes all possible angles of intersection of the image planes ($0 < \vartheta < 2\pi$, $\vartheta \neq \pi$) for each individual direction δ (cf. Fig. 3).

Thus, two further parameters δ , ϑ (ϑ in total) are required to determine in 3D the relative orientation of the two pencils of epipolar lines. These represent the direction of the second image plane relative to the first, i.e. a given 2D epipolar geometry includes, in principle, all possible directions. As discussed in the following section, knowledge of the line through the epipoles and the corresponding principal points allows establishing full relative orientation of the image planes. Then the base line o is also fixed in model space. Since a full relative orientation of an uncalibrated pair from central perspective cameras involves 11 independent parameters, the additional knowledge of one image coordinate (c , x_o or y_o) of the projective centre of each image provides full relative orientation of the pair.

• Note: Two special cases are pointed out. First, the two pencils of rays are identical, while the epipoles are not at infinity. This occurs if image planes are parallel but not coplanar (and o is not parallel to them); if image planes are coplanar but the camera constants differ; if epipoles are equidistant from the intersection of image planes. In such a case, circle c_k degenerates to a point coinciding with e_1 . For every line g , a circle through e_1 about g and on a plane normal to it is the locus of e_2 . Here, nonetheless, $\vartheta = 0$ is a valid angle referring to parallel image planes (g at infinity). Second, the epipolar lines of one image run parallel to each other, which occurs if this image is parallel to the base line o . It can be shown that, if epipole e_1 is at infinity, circle c_k degenerates to line MN (which can also be constructed).

3. PARTLY CALIBRATED IMAGES

Contrary to a typical photogrammetric approach, even in order to perform RO in its conventional sense (namely, allowing metric reconstruction) one does not need to assume already formed bundles; indeed, recovery of RO is possible together with partial camera calibration. Chang (1986) had given an early illustration of the possibility to find the IO parameters with a simultaneous adjustment of independent stereo pairs from the same camera. Faugeras et al. (1992) showed that assumption of common IO in an image pair produces two independent conditions among the elements of \mathbf{F} and the IO parameters. Hence, if certain camera elements are considered as known, partial self-calibration is feasible from a single stereo pair. By fixing the principal point, for instance, one may recover the constant of a central perspective camera even if it varies between the two views (Hartley, 1992). Non-iterative algorithms have been reported for estimating one or two camera constants from the fundamental matrix, while critical configurations have also been demonstrated (Newsam et al., 1996; Sturm, 2001; Sturm et al., 2005).

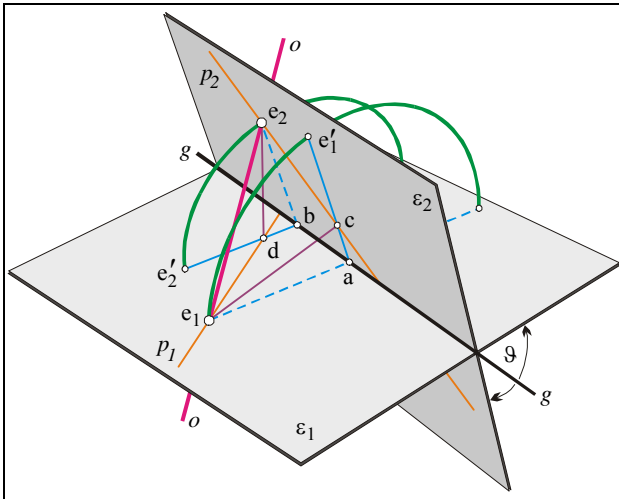


Figure 3. A possible relative position of images in 3D.

In order to address this issue here, it is referred to Fig. 3, which presents one out of the possible relative orientations of two images ε_1 , ε_2 given their line of intersection g . The rotation of an image plane about g by a change in angle ϑ does not affect the epipolar lines or the coplanarity constraint. As mentioned, such rotations move epipoles e_1 , e_2 on two parallel circles, which are normal to both image planes and their intersection g . In Fig. 3 line segments e_1a , $e_2'b$ represent the projections of the two circles on image plane ε_1 , and e_2b , $e_1'a$ are their projections on ε_2 .

Any two points on line o (e_1e_2) which joins the two epipoles can be chosen as projection centres. Camera constant and principal point corresponding to each projection centre can be determined through its normal to the respective image plane. As a consequence, for a particular angle ϑ the locus of the principal point on each image is a line through its epipole. One of them (e_1d) is constructed through the projection d of e_2 on plane ε_1 ; in a similar way, the line e_2c of the principal point may be found on ε_2 . The two right triangles e_1ac and e_2bd are then similar since their angles e_1ac and e_2bd are equal to angle ϑ of the image planes. Hence, $ac/e_1a = bd/e_2b = \cos\vartheta$. Since $e_1a = e_1'a$ and $e_2b = e_2'b$ (radii of circles) it also holds that $ac/e_1'a = bd/e_2'b = \cos\vartheta$. Consequently, the change of angle ϑ affects on both images the direction of the line through the epipole and the principal point.

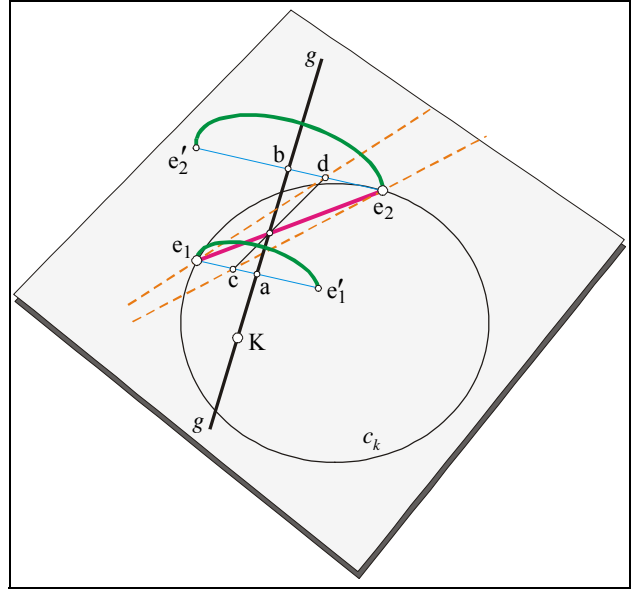


Figure 4. Inclination of plane ε_2 on ε_1 (cf. Fig. 3).

Referring now to Fig. 4, which shows the inclination on image plane ε_1 , the line e_1d of the principal point of the first image has to intersect segment e_2e_2' . Thus, for a given δ only a part of the image plane represents a valid location for the principal point. The same holds for the second image. But if the principal point of the first image, or simply the direction of line e_1d , is known, then line e_2c of the principal point of the other image is further constrained to intersect segment e_1e_1' at a specific point c such that $ac/e_1a = bd/e_2b$. From the isosceles trapezium $e_1e_1'e_2e_2'$ it can be shown that equality of these two ratios exists only when the intersection of lines cd and e_1e_2 lies on line g . To summarize, if line e_1d is known, then for every angle δ the line of the principal point of the second image as well as the angle ϑ of the two images can both be found. It is further observed that the direction of e_1d also constrains angle δ to those values which give segments e_2e_2' that can be intersected by e_1d , which means that only part of circle c_k represents now valid positions for e_2 .

If in addition to e_1d also line e_2c of the other principal point is known, the constraint that the two lines must intersect segments

e_1e_1' and e_2e_2' to equal ratios allows finding the compatible angle δ , since random selections of δ will not produce ratios ac/e_1a and bd/e_2b which are equal¹. The fact that both ratios also equal $\cos\vartheta$ finally provides the 2 missing parameters, thus allowing a full estimation of relative orientation of the two image planes and, consequently, fixing through the two epipoles the base line o in space. It is noted, however, that ϑ only establishes the angle between the image planes. Thus, $\cos\vartheta$ provides the four angles $\vartheta, \pi-\vartheta, \pi+\vartheta, 2\pi-\vartheta$, which in fact represent the four possible solutions for relative orientation².

From the above it is seen that if not only the directions of these two lines (loci of the principal points) but the principal points themselves are known, the camera constant of the two images may be found through the projections of the respective principal points onto the base line o in space. This is in agreement to the knowledge that, in general, fixed principal points allow computation of the camera constants from the fundamental matrix.

• Note: In case of coplanarity of the optical axes³ their common plane will be perpendicular to both images and their line of intersection g . In such a case the two circles of Figs. 3 and 4 will be coplanar and the two principal point lines (e_1d, e_2c) will coincide with segments e_1e_1' and e_2e_2' . In this situation none of the two ratios mentioned above can be determined and a rotation ϑ will not affect the coplanarity of the two axes. If the principal points themselves are known, for every ϑ there emerge different camera constants. Thus, coplanarity of image axes renders partial camera calibration impossible (Newsam et al., 1996). Yet, if the images are assumed as having identical camera constants, it is indeed possible to find that angle ϑ which will result to equal camera constants (Sturm, 2001). This will not hold if the two principal points are equidistant from the intersection g of image planes since then all angles ϑ produce equal values for the two camera constants. For identical camera constants this, of course, is equivalent with the equidistance of projection centres from g , also including parallelism of the optical axes, which is the critical geometry pointed out by Sturm (2001).

4. CONCLUDING REMARKS

In photogrammetric literature two distinct definitions of relative orientation of a stereopair coexist. The one presents it strictly as a separate 5-parameter orientation step following camera calibration (six parameters for two central perspective cameras) and founded on the intersection in 3D space of corresponding rays, which thus permits the metric reconstruction of object shape. A much wider view (embodied in the fundamental matrix, as elaborated in the computer vision literature) bypasses the camera calibration step and conceives relative orientation also as the 7-parameter 2D task of establishing homologue pencils of epipolar lines, which then allows object reconstruction up to a 3D projective transformation. Along with the relative position of (not known) bundles of rays which created the original image pair, the second group of parameters apparently incorporates ∞^4 combinations of interior and relative image orientations. The authors feel that photogrammetric literature needs to further scrutinize this ground between 2D and 3D epipolar geometry in a Euclidean framework.

¹ To this point, however, it is not known to the authors whether more than one solution for δ is possible.

² At this stage the criterion that reconstructed points should be in front of both cameras (Stefanovic, 1973) cannot be applied, since the principle point could be on either side of the epipole in both images.

³ If the principal points or the lines connecting them with the epipoles are known, a way to distinguish whether the optical axes are coplanar or not is to check whether these lines are also homologue epipolar lines.

This is the motivation behind the attempt made here to handle existing degrees of freedom in a more directly geometric manner and illustrate how these are constrained once partial knowledge of interior orientation is available. Besides further elaboration of the approach presented here, future tasks include similar studies of the case when only the camera constants are regarded as known and, also, of the possible image configurations if an identical interior orientation of the image pair is assumed. Further, it is intended to investigate in this framework the possibility of other factorizations of epipolar geometry with some practical use. Finally, study of overlapping image triplets in a similar manner is also within the authors' intentions.

REFERENCES

- Bender, L.U., 1971. Analytical Photogrammetry: a Collinear Theory. Final Technical Report RADC-TR-71-147, Rome Air Development Center, New York.
- Chang, B., 1986. The formulae of the relative orientation for non-metric camera. *International Archives of Photogrammetry & Remote Sensing*, 26(5):14-22.
- Faugeras, O.D., 1992. What can be seen in three dimensions with an uncalibrated stereo-rig? *European Conference on Computer Vision*, Springer, pp. 563-578.
- Faugeras, O.D., Luong Q.T., Maybank S. J., 1992. Camera self-calibration: theory and experiments. *European Conference on Computer Vision*, Springer, pp. 321-334.
- Faugeras, O.D., Luong Q.-T., 2001. *The Geometry of Multiple Images*. The MIT Press, Cambridge, Mass.
- Finsterwalder, S., 1899. Die geometrischen Grundlagen der Photogrammetrie. *Jahresbericht der Deutschen Mathem.-Vereinigung (Deutsche Gesellschaft für Photogrammetrie, Sebastian Finsterwalder zum 75. Geburtstag, Verlag Herbert Wichmann, Berlin, 1937, pp. 17-45)*.
- Hallert B., 1960. *Photogrammetry: Basic Theory and General Survey*. McGraw-Hill, New York.
- Hartley, R., 1992. Estimation of relative camera positions for uncalibrated cameras. *European Conference on Computer Vision*, Springer, pp. 579-587.
- Hartley, R., Zisserman, A., 2000. *Multiple View Geometry in Computer Vision*. Cambridge University Press.
- Khlebnikova, T., 1983. Determining relative orientation angles of oblique aerial photographs. *Mapping Science & Remote Sensing*, 21(1):95-100.
- McGlone, J.C. (ed.), 2004. *Manual of Photogrammetry*. American Society for Photogrammetry and Remote Sensing, 5th edition, Bethesda, Maryland.
- Mikhail E.M., Bethel J.S., McGlone J.C., 2001. *Introduction to Modern Photogrammetry*. John Wiley & Sons, Inc., New York.
- Newsam, G.N., Huynh, D.Q., Brooks, M.J., Pan H.P., 1996. Recovering unknown focal lengths in self-calibration: an essentially linear algorithm and degenerate configurations. *International Archives of Photogrammetry & Remote Sensing*, 31(3):575-580.
- Rinner, K., Burkhardt, R., 1972. *Photogrammetrie. Handbuch der Vermessungskunde, Band III a/3*, J.B. Metzlersche Verlagsbuchhandlung, Stuttgart.

Stefanovic, P., 1973. Relative orientation – a new approach. ITC Journal, no. 3, pp. 417-447.

Sturm, P., 2001. On focal length calibration from two views. IEEE Int. Conference on Computer Vision & Pattern Recognition, pp. 145–150.

Sturm, P., Cheng, Z.L., Chen, P.C.Y., Poo A.N., 2005. Focal length calibration from two views: method and analysis of singular cases. Computer Vision and Image Understanding, 99(1): 58-95.

Thompson, E. H., 1959. A rational algebraic formulation of the problem of relative orientation. The Photogrammetric Record, 3(14):152-159.

Wolf P.R., DeWitt B.A., 2000. Elements of Photogrammetry with Applications in GIS. McGraw-Hill, New York.