THE REFLECTED LIGHT POLARIZATION ACCOUNT AT THE SATELLITE REMOTE SENSING OF THE CLOUD COVER

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ABSTRACT:

The mathematical modelling of the polarized radiation transfer (RT) at the satellite remote sensing (SRS) of the strongly anisotropic cloudy atmosphere needs an efficient well-conditioned algorithm with accelerated solution convergence to be developed. The accelerated convergence can be achieved by the small angle approximation (SAA) subtracting from the general solution of the vectorial (polarization) RT equation (VRTE). The reminder, being a smooth function, is computed rapidly. In the current paper the SAA is determined in the frame of reference (FoR) containing the arbitrary directed sunlight rays. In contrast to the FoR containing the vertical direction of the slab the system for the coefficients of the radiation field with the coupled mutual azimuthally Fourier-expansion amplitudes is obtained. On the assumption of the continuous dependence of the angular generalized spherical function (GSF) expansion of the radiation field we use the Taylor series for the above mentioned coefficients with only two first terms left with respect to the strong anisotropy - the vectorial modified SAA of the spherical harmonics method (MSH). Some computational results are given in comparison with the single scattering approximation for thin slabs, the Monte-Carlo simulation for general case scattering and the total intensity of the polarized light beam is compared with the scalar spherical harmonics method (SHM). The algorithm for back scattered radiation computation for the SRS is given as well.

1. INTRODUCTION

It is well known that the polarization is one of the main characteristics of the radiation. Some important information such as pollution factors can be obtained for example in ecological remote sensing of the atmosphere, natural waters and the Earth as a whole using the polarization. Some satellite polarization programs such as Polder and Parasol can be mentioned to prove both the actuality of polarization SRS and inverse radiation problems solution.

The solution of the VRTE being the basis of the remote sensing is quite hard to be obtained. That is why some approximations such as single scattering underlie the satellite data processing but it is obvious that this solution is quite poor in general scattering case. We believe that more useful information can be extracted from the SRS data while using more complicated approximation.

We'll consider the Earth's atmosphere as a slab and the Sun as a plain unidirectional (PU) source of radiation as it is usual done. Besides, we'll admit an arbitrary irradiance angle and an arbitrary polarization state of the incident beam (it is the natural light for the Sun of course).

Some problems for the scalar RT (neglecting polarization) has already been mentioned in different papers (Karp, 1980): the computational time for the strong anisotropic scattering, the ill conditionality of some matrices of the VRTE solution and some others. Our aim is to create an effective method of the VRTE solution in as general case as possible and to avoid the above mentioned problems.

We'll try to base our method upon the SHM (Kuščer, 1959) improved by SAA. The PU source is described mathematically as Dirac delta function $\vec{\mathbf{L}}_0\delta(\hat{\mathbf{l}}-\hat{\mathbf{l}}_0)$ where $\hat{\mathbf{l}}=[\mu \quad \varphi]$ is a viewing direction, $\mu=\cos^{-1}\theta$, θ, φ - are the zenith and the azimuth angles respectively. $\vec{\mathbf{L}}=[\mathrm{I} \quad \mathrm{Q} \quad \mathrm{U} \quad \mathrm{V}]^{\mathrm{T}}$ is the Stokes vector for the beam defined as usual. The direction and optical depth dependence of the Stokes components are omitted somewhere for the sake of shortness, i.e. $\vec{\mathbf{L}}(\tau,\hat{\mathbf{l}})=\vec{\mathbf{L}}.$ Here and further on the upper index «→» stands for a four-elements column vector, «↔» denotes a sixteen-elements matrix, «∧» stands for a unity vector and the lower index «0» denotes a parameter of the incident beam. According to Chandrasekhar the direct light singularity can be subtracted from the general solution and then the reminder can be expanded to the spherical harmonics (SH) series as it is done for example in (Kuščer, 1959; Siewert, 2000). But for the strongly anisotropic scattering medium this reminder being a peak-function still needs a numerous expansion terms. This increases the computation time, the matrix dimensions and the instability.

We offer to present the required Stokes parameter for the diffusion polarized light field combined of two parts

$$\vec{\mathbf{L}}(\tau,\hat{\mathbf{l}})=\vec{\mathbf{L}}_{\mathrm{MSH}}(\tau,\hat{\mathbf{l}})+\vec{\mathbf{L}}_{\sim}(\tau,\hat{\mathbf{l}}), \qquad (1)$$

where the former $\vec{\mathbf{L}}_{\mathrm{MSH}}(\tau,\hat{\mathbf{l}})$ is the vectorial small angle modification of the SHM (MSH) computed in this paper and the letter $\vec{\mathbf{L}}_{\sim}(\tau,\hat{\mathbf{l}})$ is a smooth remainder (SR) which needs less expansion terms. The total time needed to compute both the MSH and the SR and matrix dimensions are less than in Chandrasekhar case. The algorithm for the SR computation is presented as well.

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2. THE VECTORIAL SPHERICAL HARMONICS METHOD

2.1 The polarization state representation

It has been already mentioned that the Stokes vector is commonly used for the RT problems. It gives the general description of any light beam which is called the SP-representation (Stokes polarization). A simple Mueller matrix multiplication $\vec{\mathbf{L}} = \vec{\mathbf{M}}\vec{\mathbf{L}}_0$ describes a transformation of the vector during its interaction with a medium. The transformation matrix for the vector parameter at the rotation of the reference plane (rotator) is known (Kuščer, 1959) as $\vec{\mathbf{R}}(\hat{\mathbf{l}}_0 \times \hat{\mathbf{l}}' \to \hat{\mathbf{l}} \times \hat{\mathbf{l}}') = \vec{\mathbf{R}}(\chi)$, where χ is the rotation angle between two planes given by cross-product $\hat{\mathbf{l}}_0 \times \hat{\mathbf{l}}' \to \hat{\mathbf{l}} \times \hat{\mathbf{l}}'$ and $\hat{\mathbf{l}}'$ is the direction of the light incidence on an elementary volume of a scattering medium. The Stokes vector transformation during the interaction with the elementary volume of a scattering medium is given by the equation

$$\vec{\mathbf{S}} = \vec{\mathbf{R}}(\hat{\mathbf{l}} \times \hat{\mathbf{l}}' \to \hat{\mathbf{l}} \times \hat{\mathbf{l}}_0)\vec{\mathbf{x}}(\hat{\mathbf{l}}, \hat{\mathbf{l}}')\vec{\mathbf{R}}(\hat{\mathbf{l}}_0 \times \hat{\mathbf{l}}' \to \hat{\mathbf{l}} \times \hat{\mathbf{l}}'), \qquad (2)$$

where $\vec{\mathbf{x}}(\hat{\mathbf{l}}, \hat{\mathbf{l}}')$ is a scattering matrix of the medium.
Using the matrix transformation

$$\vec{\mathbf{T}}_{SC} = \begin{bmatrix} 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \end{bmatrix} \qquad (3)$$

makes it possible to lead the rotation matrix to a diagonal one in the following way (Kuščer, 1959)

$$\begin{aligned} &\vec{\mathbf{R}}_{CP}(\chi) = \vec{\mathbf{T}}_{SC}\vec{\mathbf{R}}\vec{\mathbf{T}}_{SC}^{-1} = \\ &= \mathrm{Diag}\left\{\exp(+i2\chi);\ \exp(+i0\chi);\ \exp(-i0\chi);\ \exp(-i2\chi)\right\}. \end{aligned} \qquad (4)$$

The physical meaning of (3) is that the circular basis is selected for the polarization properties description. One can transform any matrix from SP-presentation to CP-presentation (Circular polarization) using (4) and any SP-vector to CP one using $\vec{\mathbf{L}}_{\mathrm{CP}} = \vec{\mathbf{T}}_{\mathrm{SC}}\vec{\mathbf{L}}_{\mathrm{SP}}$ and backward using $\vec{\mathbf{L}}_{\mathrm{SP}} = \vec{\mathbf{T}}_{\mathrm{CS}}\vec{\mathbf{L}}_{\mathrm{CP}}$. The lower index «**SC**» means «from **S**tokes basis to **C**ircular one», «CS» stands for inverse transformation and $\vec{\mathbf{T}}_{\mathrm{CS}} = \vec{\mathbf{T}}_{\mathrm{SC}}^{-1}$. Note that (3) is a complex-number transformation. Hence $\vec{\mathbf{L}}_{\mathrm{CP}}$ is a complex-number vector in spite of the fact that $\vec{\mathbf{L}}_{\mathrm{SP}}$ is the real-number energetic one.

2.2 The boundary problem for the vectorial SHM

Now let's write down the well known boundary problem for the VRTE, the PU source and a homogeneous slab

$$\begin{cases} \mu\dfrac{\partial}{\partial\tau}\vec{\mathbf{L}}(\tau, \hat{\mathbf{l}}) + \vec{\mathbf{L}}(\tau, \hat{\mathbf{l}}) = \dfrac{\Lambda}{4\pi}\oint \vec{\mathbf{S}}(\hat{\mathbf{l}}, \hat{\mathbf{l}}')\vec{\mathbf{L}}(\tau, \hat{\mathbf{l}}')d\hat{\mathbf{l}}' \\ \vec{\mathbf{L}}(0, \hat{\mathbf{l}})\Big|_{\Omega_+} = \vec{\mathbf{L}}_0\delta(\hat{\mathbf{l}} - \hat{\mathbf{l}}_0) \\ \vec{\mathbf{L}}(\tau_0, \hat{\mathbf{l}})\Big|_{\Omega_-} = \vec{\mathbf{0}}. \end{cases} \qquad (5)$$

Here Λ is single scattering albedo, τ_0 is the total optical thickness of the slab and Ω_+ and Ω_- denotes the upper and the lower half-spaces for the corresponding boundaries. All other symbols are defined above. The lower boundary condition is considered being nonreflecting, but this fact does not restrict the problem to the theoretical one only. But the medium is considered being homogeneous later on. The solution of (5) similar to the scalar case can be expanded on a series of spherical functions – generalized Legendre polynomials $\mathrm{P}^k_{m,n}(\mu)$ (Gelfand, 1963) (GLP). The index k stands for zenith expansion, m – for azimuth one and n – is the polarization index as it will be seen a little later from (8). For the sake of simplicity and in order to keep the formal analogy with the scalar case the matrix form for the GLP can be given immediately in CP-form as

$$\vec{\mathbf{P}}^k_m(\mu) = \mathrm{Diag}\left\{\mathrm{P}^k_{m,+2}(\mu); \mathrm{P}^k_{m,+0}(\mu); \mathrm{P}^k_{m,-0}(\mu); \mathrm{P}^k_{m,-2}(\mu)\right\} \qquad (6)$$

It can be seen from (6) that polarization index n takes on the values $+2, +0, -0, -2$ with respect to exponential powers in (4) for CP and shifted values $+0, +2, -2, -0$ for SP-case. The definition, properties, recurrent formulas and the addition theorem

$$\begin{aligned} &\exp(-ir\chi)\mathrm{P}^k_{r,s}\left(\hat{\mathbf{l}}\hat{\mathbf{l}}'\right)\exp(+is\chi') = \\ &= \sum_{m=-k}^{k}(-1)^m \mathrm{P}^k_{r,m}(\mu)\mathrm{P}^k_{m,s}(\mu')\exp\left(im(\varphi - \varphi')\right) \end{aligned} \qquad (7)$$

(where r and s runs through the values $+2, +0, -0, -2$ independently) for the GLP can be found in (Gelfand, 1963). One can see that rotation angles are included in the left-hand side of (7). The GLP expansions for the required Stokes vector and the scattering matrix are

$$\begin{aligned} &\vec{\mathbf{L}}(\tau, \hat{\mathbf{l}}) = \sum_{m=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\vec{\mathbf{P}}^k_m(\mu)\vec{\mathbf{f}}^k_m(\tau)\exp(im\varphi) \\ &\left[\vec{\mathbf{x}}(\tau, \hat{\mathbf{l}}\hat{\mathbf{l}}')\right]_{rs} = \sum_{k=0}^{\infty}(2k+1)x^k_{rs}(\tau)\mathrm{P}^k_{rs}(\hat{\mathbf{l}}\hat{\mathbf{l}}'), \end{aligned} \qquad (8)$$

both for the CP-presentation. Using the expansions (8), the addition theorem (7), some recurrence formulas from (Gelfand, 1963) and the CP-diagonalized rotator (4) allows to evaluate the scattering integral in (5) (Kuščer, 1959) and to obtain the system of differential equations for the expansion coefficients in (8):

$$\frac{1}{2k+1}\frac{\partial}{\partial\tau}\left[\vec{\mathbf{A}}_m^{k+1}\vec{\mathbf{f}}_m^{k+1}(\tau)+\vec{\mathbf{B}}_m^k\vec{\mathbf{f}}_m^k(\tau)+\vec{\mathbf{A}}_m^k\vec{\mathbf{f}}_m^{k-1}(\tau)\right]+ +\left[\vec{\mathbf{1}}-\Lambda\vec{\mathbf{x}}^k\right]\vec{\mathbf{f}}_m^k(\tau)=\vec{\mathbf{0}}, \tag{9}$$

where

$$\left[\vec{\mathbf{A}}_m^k\right]_{rs}=\frac{1}{k}\sqrt{\left(k^2-m^2\right)\left(k^2-s^2\right)}\delta_{rs};$$

$$\left[\vec{\mathbf{B}}_m^k\right]_{rs}=\frac{ms}{k\left(k+1\right)}\left(2k+1\right)\delta_{rs}.$$

and δ_{rs} is the Kronecker symbol. The matrix form for the bottom boundary condition in the case of nonzero Lambertian reflection is given in (Siewert, 2000). The above mentioned system (9) is written for CP-presentation and for the traditional FoR: μ is counted off from the normal to the slab which is directed downwards (to the Earth's surface). The complex values in (9) prevent using some effective methods worked out for the scalar case. Therefore it is an expedient action to apply back matrix transformation of (9) to the real SP-representation after the scattering integral had been evaluated in CP.

3. THE VECTORIAL SPHERICAL HARMONICS MODIFICATION

3.1 The small angle approximation for polarization case

The phase function (a [1,1]-positioned element of $\vec{\mathbf{x}}(\hat{\mathbf{l}},\hat{\mathbf{l}}')$) of different real medium of RT such as clouds and natural waters are strongly anisitropical functions along the direction of light incidence. This allows to assume that the angles of scattering are small and to simplify the RTE. The obtained small angle approximation (SAA) contains the δ-singularity of the direct light for the PU-source. This approximation can be used both to compute forward scattered radiation and for the acceleration of convergence of the RTE solution (Budak, 2004a) by using the algorithm based on the idea (1). Now we'll describe the SAA for the polarization case.

We'll change the FoR and will further on count off the sight angle ν (instead of μ) from the direction of the light incidence $\hat{\mathbf{l}}_0$ (Budak, 2004b) and the azimuth angle ψ (instead of φ) in the plane perpendicular to $\hat{\mathbf{l}}_0$. This leads to the transformation of the differential term in (5) (first item in the left-hand side of the VRTE). One can use (Gelfand, 1963) to derive some recurrence formula for the GLP for the specified case and obtain the system similar to (9) but a little more complicated. We'll give the common form of this system:

$$\frac{1}{2k+1}\frac{\partial}{\partial\tau}\Big\{\mu_0\left[\vec{\mathbf{A}}_m^{k+1}\vec{\mathbf{f}}_m^{k+1}(\tau)+\vec{\mathbf{B}}_m^k\vec{\mathbf{f}}_m^k(\tau)+\vec{\mathbf{A}}_m^k\vec{\mathbf{f}}_m^{k-1}(\tau)\right]+ +\frac{i}{2}\sqrt{1-\mu_0^2}\left[\vec{\mathbf{h}}_1^n\vec{\mathbf{f}}_{m-1}^{k-1}(\tau)+\vec{\mathbf{h}}_2^n\vec{\mathbf{f}}_{m-1}^k(\tau)+\vec{\mathbf{h}}_3^n\vec{\mathbf{f}}_{m-1}^{k+1}(\tau)+ +\vec{\mathbf{h}}_4^n\vec{\mathbf{f}}_{m+1}^{k-1}(\tau)+\vec{\mathbf{h}}_5^n\vec{\mathbf{f}}_{m+1}^k(\tau)+\vec{\mathbf{h}}_6^n\vec{\mathbf{f}}_{m+1}^{k+1}(\tau)\right]\Big\}+ +\left(\vec{\mathbf{1}}-\Lambda\vec{\mathbf{x}}^k\right)\vec{\mathbf{f}}_m^k(\tau)=\vec{\mathbf{0}}. \tag{10}$$

The expansion coefficients of some scattering matrices $\vec{\mathbf{x}}^k$ are known. Following (Astakhov, 1994) we'll admit a continuous dependence of Stokes expansion coefficients $\vec{\mathbf{f}}_m^k(\tau)\to\vec{\mathbf{f}}^m(\tau,k)$ representing the spatial or angle-dependent spectrum. In consequence of strong anisotropy of $\vec{\mathbf{L}}(\tau,\hat{\mathbf{l}})$ its spectrum will be a smooth function and using a Taylor expansion of $\vec{\mathbf{f}}^m(\tau,k)$ we'll cut it down to two terms:

$$\vec{\mathbf{f}}_{k\pm1}^m(\tau)=\vec{\mathbf{f}}^m(\tau,k\pm1)\approx\vec{\mathbf{f}}^m(\tau,k)\pm\frac{\partial\vec{\mathbf{f}}^m(\tau,k)}{\partial k}. \tag{11}$$

Besides the anisotropy of the medium allows to write $k >> m > n$ and to reconstruct matrices $\vec{\mathbf{h}}_j^n$ in (10). Having applied the above mentioned approximation to (10) we'll be able to follow two ways described below.

3.2 General MSH

According to the scalar solution described in (Budak, 2004b) we can imply a special vector-function $\vec{\boldsymbol{\omega}}(\tau,\psi,\kappa)$ where $\kappa=\sqrt{k(k+1)}$ in the following way:

$$\begin{aligned}\vec{\boldsymbol{\omega}}(\tau,\psi,\kappa)&=\sum_{m=-\infty}^{\infty}\vec{\mathbf{f}}^m(\tau,\kappa)\exp\left(im\psi\right)\\ \vec{\mathbf{f}}^m(\tau,\kappa)&=\frac{1}{2\pi}\int_0^{2\pi}\vec{\boldsymbol{\omega}}(\tau,\psi,\kappa)\exp\left(-im\psi\right)d\psi\end{aligned} \tag{12}$$

After simplification of (10) and using (12) we can obtain the equation (i is a complex unit)

$$\frac{\partial}{\partial\tau}\left[\mu_0-i(\hat{\mathbf{l}}_{0\perp},\nabla_{\boldsymbol{\kappa}})\right]\vec{\boldsymbol{\omega}}(\tau,\psi,\kappa)=-\left(\vec{\mathbf{1}}-\Lambda\vec{\mathbf{x}}^k\right)\vec{\boldsymbol{\omega}}(\tau,\psi,\kappa). \tag{13}$$

Let's reassign $\vec{\boldsymbol{\omega}}(\tau,\psi,\kappa)\to\vec{\boldsymbol{\omega}}(\tau,\hat{\mathbf{l}}_{0\perp},\kappa)$ where $\hat{\mathbf{l}}_{0\perp}$ is the projection of $\hat{\mathbf{l}}_0$ upon the plane of ψ and $(\hat{\mathbf{l}}_{0\perp},\nabla_{\boldsymbol{\kappa}})$ is the directional derivative through $\hat{\mathbf{l}}_{0\perp}$ for simplicity. We'll seek the solution in the following kind

$$\vec{\boldsymbol{\omega}}(\tau,\hat{\mathbf{l}}_{0\perp},\kappa)=\exp\left(-\frac{\tau}{\mu_0}\left(\vec{\mathbf{1}}-\Lambda\vec{\mathbf{W}}\left(\hat{\mathbf{l}}_{0\perp},\kappa\right)\right)\right)\vec{\boldsymbol{\omega}}_0, \tag{14}$$

where $\vec{\boldsymbol{\omega}}_0=\vec{\boldsymbol{\omega}}(0,\hat{\mathbf{l}}_{0\perp},\kappa)$ is defined by the top boundary condition, and we'll derive the differential equation for $\vec{\mathbf{W}}$ as following

$$\vec{\mathbf{W}}\left(\hat{\mathbf{l}}_{0\perp},\kappa\right)-\frac{i}{\mu_0}(\hat{\mathbf{l}}_{0\perp},\nabla_{\boldsymbol{\kappa}})\vec{\mathbf{W}}\left(\hat{\mathbf{l}}_{0\perp},\kappa\right)=\vec{\mathbf{x}}^k(\tau) \qquad (15)$$

This equation can be solved by the integration along $\hat{\mathbf{l}}_{0\perp}$ with $\boldsymbol{\kappa}\to\boldsymbol{\kappa}+\zeta\hat{\mathbf{l}}_{0\perp}$ where ζ is scale parameter and $(\hat{\mathbf{l}}_{0\perp},\nabla_{\boldsymbol{\kappa}})=\sin\theta_0\frac{d}{d\zeta}$. Having omitted the intermediate evaluations and simplifications we'll give the final equation for $\vec{\mathbf{W}}\left(\hat{\mathbf{l}}_{0\perp},\kappa\right)$ similar to the scalar case (Budak, 2004b)

$$\vec{\mathbf{W}}=\int_{\zeta_0}^{0} i\frac{\mu_0}{\sqrt{1-\mu_0{}^2}}\vec{\mathbf{x}}(\left|\boldsymbol{\kappa}+\zeta\hat{\mathbf{l}}_{0\perp}\right|)\exp\left(i\frac{\mu_0\zeta}{\sqrt{1-\mu_0{}^2}}\vec{\mathbf{1}}\right)d\zeta, \qquad (16)$$

which can be combined with (14) and (12) to obtain the expansion coefficients $\vec{\mathbf{f}}_m^k$ and the required solution (8). The integration in (16) must be fulfilled numerically where for the Henyey-Greenstein phase function one can write

$$x(\left|\boldsymbol{\kappa}+\zeta\hat{\mathbf{l}}_{0\perp}\right|)=\exp\left(\sqrt{\kappa^2+2\zeta\kappa\sqrt{1-\mu_0^2}\cos\psi+\zeta^2(1-\mu_0^2)}\ln g\right),$$

where g is the Henyey-Greenstein phase function parameter. So the general form of the MSH is made up.

3.3 The vectorial small angle modification

Another way is to admit $\mu_0\to 1$ in (10). We'll let $\sqrt{1-\mu_0^2}\to 0$ but we'll live the term containing μ_0 unchanged and obtain

$$\frac{1}{2k+1}\frac{\partial}{\partial\tau}\left\{\mu_0\left[\vec{\mathbf{A}}_m^{k+1}\vec{\mathbf{f}}_m^{k+1}(\tau)+\vec{\mathbf{B}}_m^k\vec{\mathbf{f}}_m^k(\tau)+\vec{\mathbf{A}}_m^k\vec{\mathbf{f}}_m^{k-1}(\tau)\right]\right\}=$$
$$+\left(\vec{\mathbf{1}}-\Lambda\vec{\mathbf{x}}^k\right)\vec{\mathbf{f}}_m^k(\tau)=\vec{\mathbf{0}}. \qquad (17)$$

from (10). Having applied all approximations mentioned in section 3.1 including (11) we obtain for the homogeneous medium

$$\vec{\mathbf{f}}^m(\tau,k)=\exp\left\{-\frac{\tau}{\mu_0}\left(\vec{\mathbf{1}}-\Lambda\vec{\mathbf{x}}^k\right)\right\}\vec{\mathbf{f}}^m(0,k), \qquad (18)$$

where $\vec{\mathbf{f}}^m(0,k)$ is defined by the top boundary condition, which can be written as (Budak, 2005)

$$\vec{\mathbf{L}}_{SP}^0(\tau=0,\hat{\mathbf{l}})=L\begin{bmatrix}1\\ p\cos2(\varphi-\varphi_0)\\ -p\sin2(\varphi-\varphi_0)\\ q\end{bmatrix}\delta(\hat{\mathbf{l}}-\hat{\mathbf{l}}_0) \qquad (19)$$

Here p is the linear polarization degree and q is the ellipticity of the incident light. Combining (19), (18) and first equation from (8) we obtain after some simplifications (using Waterloo® Maple) and transformation to SP-presentation

$$\vec{\mathbf{L}}_{SP}(\tau,\nu,\varphi)=L\sum_{k=0}^{\infty}\frac{2k+1}{2}e^{-\omega_k\tau}\left\{\begin{bmatrix}W_k(\tau) & \cdot\mathrm{P}_{0,0}^k(\nu)\\ U_k(\tau) & \cdot\mathrm{P}_{2,0}^k(\nu)\\ -qu_k(\tau) & \cdot\mathrm{P}_{2,0}^k(\nu)\\ qw_k(\tau) & \cdot\mathrm{P}_{0,0}^k(\nu)\end{bmatrix}+\right.$$
$$\left.+p\begin{bmatrix}U_k(\tau)\cdot\mathrm{P}_{2,0}^k(\nu) & \cdot\cos2\varphi'\\ W_k(\tau)\cdot\mathrm{R}_2^k(\nu)+w_k(\tau)\cdot\mathrm{T}_2^k(\nu) & \cdot\cos2\varphi'\\ -W_k(\tau)\cdot\mathrm{T}_2^k(\nu)-w_k(\tau)\cdot\mathrm{R}_2^k(\nu) & \cdot\sin2\varphi'\\ -u_k(\tau)\cdot\mathrm{P}_{2,0}^k(\nu) & \cdot\sin2\varphi'\end{bmatrix}\right\}, \qquad (20)$$

where $\varphi'=\varphi-\varphi_0$, $\mathrm{R}_2^k(\nu)=\frac{1}{2}\left(\mathrm{P}_{2,2}^k(\nu)+\mathrm{P}_{2,-2}^k(\nu)\right)$, $\mathrm{T}_2^k(\nu)=\frac{1}{2}\left(\mathrm{P}_{2,2}^k(\nu)-\mathrm{P}_{2,-2}^k(\nu)\right)$ and W_k, U_k, w_k, u_k, ω_k are some functions of eigenvectors of $\vec{\mathbf{x}}^k$ (Budak, 2005). One can use various numerical algorithms to evaluate the exponential matrix in (18) (for example by means of The Mathworks® Matlab function *expm*). It won't take much time in comparison with computing using (20) and it will make the computational program a little simpler but one will not be able to obtain the analytical solution such as (20) and to see how different parameters influence to the desirable solution without making numerous computations. The obtained system (20) is the vectorial small angle modification (VSAM) of the SHM.

4. THE BACKSCATTERED RADIATION

Let's substitute (1) into the VRTE (5). We'll obtain

$$\mu\frac{\partial}{\partial\tau}\vec{\mathbf{L}}_{\sim}(\tau,\hat{\mathbf{l}})+\vec{\mathbf{L}}_{\sim}(\tau,\hat{\mathbf{l}})=\frac{\Lambda}{4\pi}\oint\vec{\mathbf{S}}(\hat{\mathbf{l}},\hat{\mathbf{l}}')\vec{\mathbf{L}}_{\sim}(\tau,\hat{\mathbf{l}}')d\hat{\mathbf{l}}'+\Delta(\tau,\hat{\mathbf{l}}), \quad (21)$$

where the source function is

$$\Delta(\tau,\hat{\mathbf{l}})=-\mu\frac{\partial}{\partial\tau}\vec{\mathbf{L}}_{\mathrm{MSH}}(\tau,\hat{\mathbf{l}})-\vec{\mathbf{L}}_{\mathrm{MSH}}(\tau,\hat{\mathbf{l}})+\vec{\mathbf{I}}_S \qquad (22)$$

$$\vec{\mathbf{I}}_S=\frac{\Lambda}{4\pi}\oint\vec{\mathbf{S}}(\hat{\mathbf{l}},\hat{\mathbf{l}}')\vec{\mathbf{L}}_{\mathrm{MSH}}(\tau,\hat{\mathbf{l}}')d\hat{\mathbf{l}}'. \qquad (23)$$

Applying (3) to (23) we convert it to CP-presentation, expand the scattering matrix and the Stokes parameter in series on the GLP, use (Siewert, 2000) to determine the following matrices

$$\vec{\phi}_1(\varphi)=\mathrm{diag}\left\{\cos\varphi,\cos\varphi,\sin\varphi,\sin\varphi\right\},$$
$$\vec{\phi}_2(\varphi)=\mathrm{diag}\left\{-\sin\varphi,-\sin\varphi,\cos\varphi,\cos\varphi\right\},$$
$$\vec{\mathbf{D}}_1=\mathrm{diag}\left\{1,1,0,0\right\},\ \vec{\mathbf{D}}_2=\mathrm{diag}\left\{0,0,-1,-1\right\},$$
$$\vec{\mathbf{\Pi}}_k^m(\mu)=\begin{bmatrix}\mathrm{P}_{2m}^k(\mu) & 0 & 0 & 0\\ 0 & \mathrm{R}_{2m}^k(\mu) & \mathrm{T}_{2m}^k(\mu) & 0\\ 0 & \mathrm{T}_{2m}^k(\mu) & \mathrm{R}_{2m}^k(\mu) & 0\\ 0 & 0 & 0 & \mathrm{P}_{2m}^k(\mu)\end{bmatrix} \tag{24}$$

and after simplifications (Budak, 2005) we can obtain the scattering integral ($\vec{\mathbf{L}}$ is used to designate both $\vec{\mathbf{L}}_{\mathrm{MSH}}(\tau,\hat{\mathbf{l}})$ and $\vec{\mathbf{L}}_{\sim}(\tau,\hat{\mathbf{l}})$).

$$\vec{\mathbf{I}}_S=\frac{\Lambda}{4\pi}\oint\Bigg[\sum_{k=0}^{\infty}\sum_{m=0}^{k}(2k+1)(2-\delta_{0m})\Big(\vec{\phi}_1(m(\varphi-\varphi'))\vec{\mathbf{A}}\vec{\mathbf{D}}_1+$$
$$+\vec{\phi}_2(m(\varphi-\varphi'))\vec{\mathbf{A}}\vec{\mathbf{D}}_2\Big)\Bigg]\vec{\mathbf{L}}(\tau,\hat{\mathbf{l}}')d\hat{\mathbf{l}}', \tag{25}$$

Here $\vec{\mathbf{A}}_k^m(\mu,\mu')=\vec{\mathbf{\Pi}}_k^m(\mu)\vec{\boldsymbol{\chi}}_k\vec{\mathbf{\Pi}}_k^m(\mu'),\ \vec{\boldsymbol{\chi}}_k=\vec{\mathbf{T}}_{SC}\vec{\mathbf{x}}_k\vec{\mathbf{T}}_{CS}$. In order to obtain the source function (22) we convert (20) back to CP, and change the FoR for the sight angle counted off from boundary surface normal instead of the irradiance direction $\hat{\mathbf{l}}_0$, i.e. we begin to use μ instead of ν again (see section 3.1). We combine this result (Budak, 2005) with (22), (23), (25) and give it the following form

$$\Delta(\tau,\mu,\varphi)=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{2k+1}{2}\Big(\vec{\phi}_1(m\varphi)\vec{\mathbf{\Pi}}_k^m(\mu)\vec{\mathbf{p}}_k(\tau)\vec{\mathbf{\Pi}}_k^m(\mu_0)\vec{\mathbf{D}}_1+$$
$$+\vec{\phi}_2(m\varphi)\vec{\mathbf{\Pi}}_k^m(\mu)\vec{\mathbf{p}}_k(\tau)\vec{\mathbf{\Pi}}_k^m(\mu_0)\vec{\mathbf{D}}_2\Big)\Big]\vec{\mathbf{L}}_0\exp\left(-\frac{\tau}{\mu_0}\right), \tag{26}$$

where $\vec{\mathbf{p}}_k(\tau)$ is the transfer matrix GLP expansion coefficients in SP representation.
If we assume the sufficiency of smoothness we'll be able to compute $\vec{\mathbf{L}}_{\sim}$ as the diffusion component (Siewert, 2000) in the following form

$$\vec{\mathbf{L}}_{\sim}=\sum_{m=0}^{\infty}\Big[\vec{\phi}_1(m\varphi)\vec{\mathbf{L}}_{\sim1}^m(\tau,\mu)+\vec{\phi}_2(m\varphi)\vec{\mathbf{L}}_{\sim2}^m(\tau,\mu)\Big], \tag{27}$$

and from two integral equations (j = 1,2) which have already been integrated over φ

$$\mu\frac{\partial}{\partial\tau}\vec{\mathbf{L}}_{\sim j}^m(\tau,\mu)+\vec{\mathbf{L}}_{\sim j}^m(\tau,\mu)=$$

$$=\frac{\Lambda}{2}\sum_{k=0}^{\infty}(2k+1)\int_{-1}^{1}\vec{\mathbf{A}}_k^m(\mu,\mu')\vec{\mathbf{L}}_{\sim j}^m(\tau,\mu')d\mu'+\vec{\mathbf{\Delta}}_j(\tau,\mu), \tag{28}$$

where $\vec{\mathbf{\Delta}}_j(\tau,\mu)=\sum_{k=0}^{\infty}\frac{2k+1}{2}\vec{\mathbf{\Pi}}_k^m(\mu)\vec{\mathbf{p}}_k(\tau)\vec{\mathbf{\Pi}}_k^m(\mu_0)\vec{\mathbf{D}}_j$, satisfying two boundary conditions $\vec{\mathbf{L}}_{\sim j}^m(0,\mu)\big|_{\mu\geq0}=0$ and $\vec{\mathbf{L}}_{\sim j}^m(\tau_0,\mu)\big|_{\mu\leq0}=-\Big[\vec{\mathbf{L}}_{\mathrm{MSH}}^m(\tau_0,\mu)\Big]_j$. The obtained integral equations admit the solution using discrete ordinate method. Thus the problem is reduced to the system of ordinary differential equations and can be solved numerically after the truncation from infinity to a finite number of equations.

5. DISCUSSION

Numerous approximations made in order to evaluate (20) produces a lot of difficulties in verifying the result. The only way is to compare the result with some others obtained by independent methods. We'll give some calculations compared with different methods on the figures 1-4.

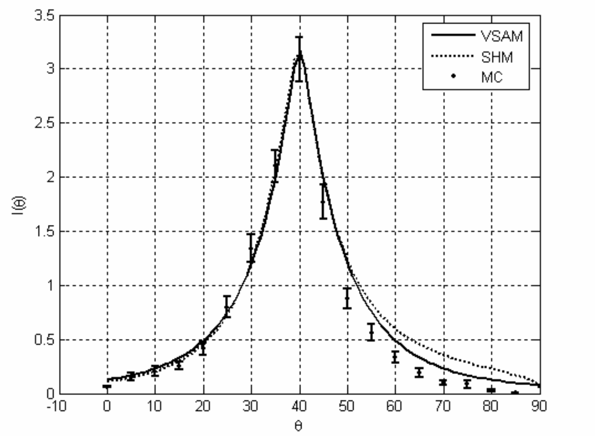

Figure 1. The zenith dependence of I component of $\vec{\mathbf{L}}$. θ_0= 40°, τ = 5, g=0.95, Λ=0.99. Solid line – VSAM, dot line – MSH (300 terms for both zenith and azimuth expansions), errorbar – MC (for 1000 rays - average of distribution and the standard deviation). The incident light is natural. The medium is nonpolarized $P_m = Q_m = 0$.

The following abbreviations are used: VSAM – the vectorial small angle modification using the obtained system (20); MC – the Monte-Carlo simulation modified by the local estimation calculation; SS – single scattering approximation; SHM – scalar spherical harmonics method (Karp, 1980).
The polarization degree is defined as $p=\mathrm{Q}(\theta)/\mathrm{I}(\theta)$ and the aerosol Henyey-Greenstein scattering matrix is admitted:

$$\vec{\mathbf{x}}_{\mathrm{SP}}(\mu)=x_{11}\begin{bmatrix}1 & P_m-\mu^2P_m & 0 & 0\\ P_m-\mu^2P_m & 0.25\left(1+\mu\right)^2 & 0 & 0\\ 0 & 0 & 0.25\left(1+\mu\right)^2 & Q_m-\mu^2Q_m\\ 0 & 0 & -Q_m+\mu^2Q_m & 1\end{bmatrix},$$

where $x_{11} = \dfrac{1-g^2}{\left(1+g^2-2g\mu\right)^{3/2}}$, and P_m, Q_m are the polarization degree and the ellipticity for the single scattering act respectively; they submit to some conditions (Hovenier, 1996). Other parameters are given in the inscriptions for the corresponding figures. We must note that Rayleigh scattering matrix and the exact solution given by Chandrasekhar can be used to verify the method under consideration.

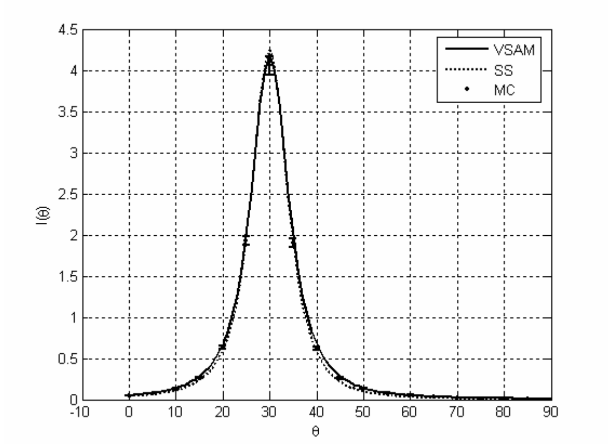

Figure 2. The zenith dependence of I component of $\vec{\mathbf{L}}$. P_m = 0.5, Q_m = 0.2. θ_o= 30°, τ = 0.5, g=0.9, Λ=0.2. Solid line – VSAM, dot line – SS, errorbar - MC (for 5000 rays - average of distribution and the standard deviation). The incident light is polarized - [1 1 0 0]

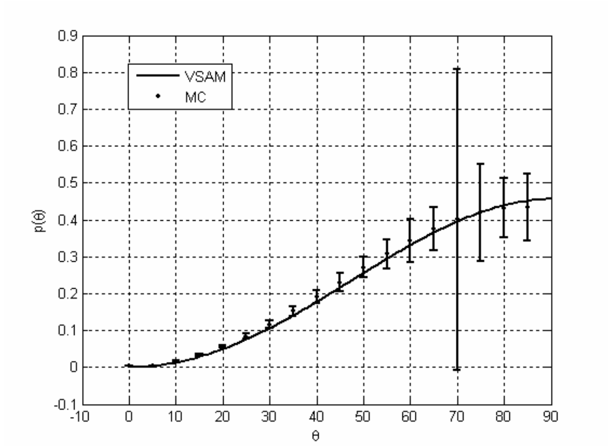

Figure 3. The zenith polarization degree *p* dependence. $P_m = Q_m$ = 0.5. θ_o= 0°, τ = 1, g=0.9, Λ=0.8. Solid line – VSAM, errorbar - MC (for 1000 rays - average of distribution and the standard deviation which can be decreased by increasing the number of rays). The incident light is natural.

6. CONCLUSION

As it has been shown our method gives reliable results in comparison with some others. The VSAM neglects the back scattered radiation; the latter must be computed separately using section 4 and a smooth function which can easily be computed appears. But the calculation speed of the VSAM for the case of fig.1 for instance is 30 times better than scalar SHM or polarized Monte-Carlo simulation for the forward hemisphere (directed to the Earth). The VSAM can be modified using section 3.2 in order to include some known polarization effects. The VSHM (20) enlarged to the general solution represents the accelerated method of the VRTE solution in analytical form.

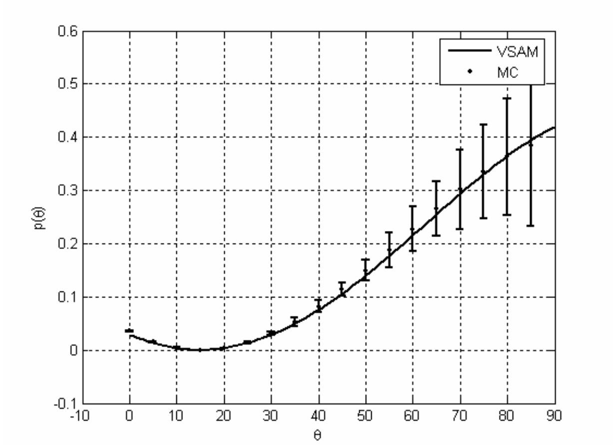

Figure 4. The zenith polarization degree *p* dependence. P_m = 0.5, Q_m = 0.2. θ_o= 15°, τ = 1, g=0.9, Λ=0.8. Solid line – VSAM, errorbar - MC (for 5000 rays - average of distribution and the standard deviation). The incident light is elliptically polarized [1 0 0 1].

REFERENCES

Astakhov, I., 1994. Solution of the vector radiative transfer equation in the small-angle approximation of the spherical harmonics method. *Atm.Opt.*, 7, pp. 398-402.

Budak, V., 2004a. Convergence acceleration of a spherical harmonics method at strongly anisotropic scattering. *Optics of an atmosphere and ocean*, 17(1), pp. 36-41.

Budak, V., 2004b. About the precision and application range of the small angle approximation in the theory of radiative transfer. Optics of an atmosphere and ocean, 17(12), pp. 1 - 7.

Budak, V., 2005. Mathematical model of the polarized light reflection by the turbid medium slab with an anisotropic scattering. *Proc. SPIE*, 5888, pp.19-1 – 19-8.

Gelfand, I., 1963. *Representations of the rotation and Lorentz groups and their applications*. Pergamon Press, Oxford, 1963.

Hovenier, J., 1996. Testing scattering matrices: a compendium of recipes. *JQSRT*, 55(5), pp. 649 – 661.

Karp, A., 1980. Radiative transfer through an arbitrary thick scattering atmos-phere. *JQSRT*, 24(5), pp. 391- 406.

Kuščer, I., 1959. Matrix Formalism in the Theory of Diffusion of Light. *Optica Acta*, 6(1), pp. 42-51.

Siewert, C., 2000. A discrete-ordinates solution for radiative-transfer models that include polarization effects. *JQSRT*, 64, pp. 227-254.

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