APPROXIMATE GEOMETRIC REASONING WITH EXTENDED GEOGRAPHIC OBJECTS

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ABSTRACT:

The article presents a conceptual framework for formal geometric reasoning with extended objects in the context of vernacular geography. Vernacular geography is concerned with place names and their relations as they are used in people's everyday vernacular language. Commonly used place names include names of land formations, landmarks, woods, water bodies or streets. Unlike single points that are given in Cartesian coordinates, these geographic entities are extended in space and often vaguely defined. Nevertheless people perform spatial reasoning with extended geographic entities "as if they were points": Expressions like "Prague lies half-way between Vienna and Berlin" or "The apartment is located quite between a train station, a tram stop, and a bus stop" involve not only topological relations, but also approximate geometric constructions that use extended geographic entities in the role of points. With the rise of ubiquitous computing, the ability to represent and query textual descriptions of spatial configurations in a GIS becomes increasingly important. To achieve this, it is necessary to formalize topologic and geometric reasoning with extended and vaguely defined objects. While much research has been done on topological reasoning with extended objects, geometric reasoning with extended objects has rarely been addressed.

The paper describes difficulties that arise from approximate geometric reasoning with extended objects and proposes to use a fuzzified version of David Hilbert's axiomatic logical calculus for Euclidean geometry as a way to cope with these difficulties. Based on the idea that extended objects may be seen as location constraints to coordinate points, the geometric primitives *point*, *line*, *incidence* and *equality* are interpreted as fuzzy predicates of a first order language. An additional predicate for the "distinctness" of pointlike objects is added. We confine ourselves to crisp extended objects like buildings or areas with official boundary definitions; vaguely defined geographic entities like mountains or places such as "downtown" are excluded in this paper. A fuzzification of the axioms of incidence geometry is given, which is based on the proposed fuzzy predicates. Rational Pavelka Logic is discussed as a reasoning system for a geometry of extended objects: Once a model of Euclidean geometry is found, which is based on Rational Pavelka Logic, worst-case values for the ill-posedness or well-posedness of a geometric construction can be derived. Reasoning with Rational Pavelka Logic has the advantage of being computationally less expensive and thus faster than a detailed analysis of a given spatial constellation.

1. INTRODUCTION

1.1 Spatial analysis with extended objects

In vernacular speech, place names and landmarks are often used to describe the approximate location of geographic entities. For example, the statement "The apartment is located between Vienna Western station, tram stop Stollgasse and bus stop Zieglergasse" is a textual description of the apartment's location and might be found in an advertisement. With the rise of ubiquitous computing, the automation of spatial reasoning calculi that can deal with textual descriptions and approximate location information becomes increasingly important. The simplest version of approximate location information is a crisp extended region, which can be seen as a constraint to the space an object possibly or actually occupies (Gerla, 2008). Up to date, geographic information systems (GIS) have the ability to perform topological reasoning with extended geographic objects (e.g. Dilo, 2006). Yet, the capability of geometric reasoning with extended objects is still missing. The aim of the present work is to lay a foundation for geometric reasoning with extended objects that is usable in GIS.

As an example of a geometric construction with extended objects consider again the above statement "The apartment is located between Vienna Western station, tram stop Stollgasse and bus stop Zieglergasse" and suppose it is a GIS query with the goal to represent the approximate location of the apartment in a map. Suppose the train station, the tram stop and the bus stop are known and represented in the GIS by polygons, whereas the location of the apartment is unknown.

A heuristic solution to the above problem could be to represent the three landmarks by their centroids, construct a triangle from the three coordinate points, calculate the centroid of the resulting triangle and output it as the approximate location of the apartment (Figure 1).

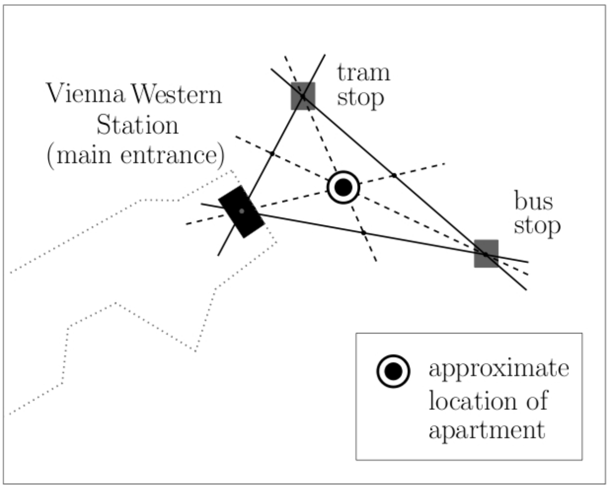

Figure 1. Three extended objects: a train station, a tram stop and a bus stop. The approximate location of the apartment is derived heuristically from a textual description.

The solution usually works fine if performed by an individual, who checks if the problem statement makes sense: "Are the involved objects approximately of the same size or do the sizes differ too much?", "Can the distances between the involved objects be displayed in the same map scale?", "Is it possible to determine an approximate line from any two input objects or is, e.g., one of them enclosing the other?", etc. If the process is automated, a calculus is needed that decides on the ill-posedness or well-posedness of the configuration.

As an example of an ill-posed problem consider the case that the polygon representing "Vienna Western Station" as stored in the GIS comprises not only the station's main entrance – which is the intended meaning of the textual description –, but with it the whole rail yard of the station. Figure 2 sketches the resulting geometrical configuration: The three extended objects that are used as input to the heuristic differ too much in their sizes to allow a meaningful result in the given context. The reason for the heuristic to fail is that the centroids, being Cartesian coordinate points, do not take into account the spatial extent of the involved polygons and provide a too rough approximation of the objects in the given geometric context.

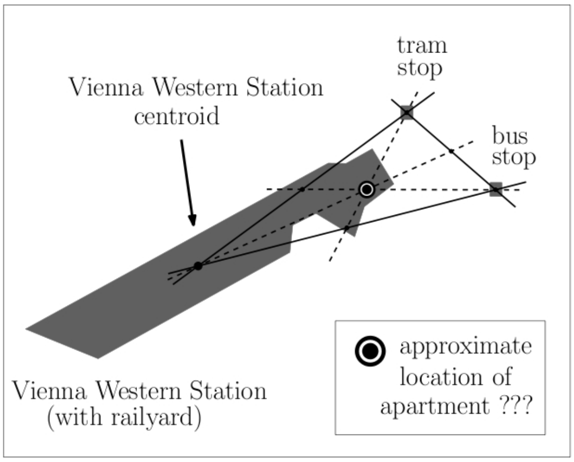

Figure 2. Ill-defined constellation of extended objects: The applied heuristic fails in the given geometric context.

The present paper proposes to tackle this problem by applying a geometry that takes the extended objects themselves as geometric primitives: a geometry of "extended points" and "extended lines" is proposed. Figure 3 shows an ad-hoc example of a construction process involving extended primitives. We show that the question of a geometric construction query being well-posed or ill-posed in the context of a specific geometric constellation is a matter of degree. Fuzzy approximate reasoning provides an instrument to define a measure of well-posedness of a geometric query and it's value can be derived for any specific geometric constellation in question. As a consequence, it is possible to suppress an automatically generated output in case the problem statement does not make sense from a geometrical point of view. The fuzzy approximate calculus proposed is Rational Pavelka logic, based on a fuzzyfication of David Hilbert's axiom system for Euclidean geometry. Since most tests and operations for spatial analysis in a vector based GIS are based on the algebra of Cartesian coordinate geometry, and thus on the axioms of Euclidean geometry, a fuzzyfication of the Euclidean axiomatic system provides an extension of the Cartesian algebra rather than a new calculus. Existing algorithms can be reused. As an illustration of the framework, we look at the axioms of incidence geometry, which is a subset of the Euclidean axiomatic system.

The remainder of the article is structured as follows: Chapter 2 briefly introduces the incidence axioms of Hilbert's axiomatic system for Euclidean geometry; Examples of possible interpretations of extended geometric primitives are given and arising problems are illustrated and formalized. In chapter 3 fuzzy predicates for geometric primitives are defined and an axiomatization of incidence geometry on the basis of these primitives is proposed. Rational Pavelka Logic is discussed as a possibility to formalize approximate deduction based on extended primitives. The article concludes with a discussion and with an outlook to further work.

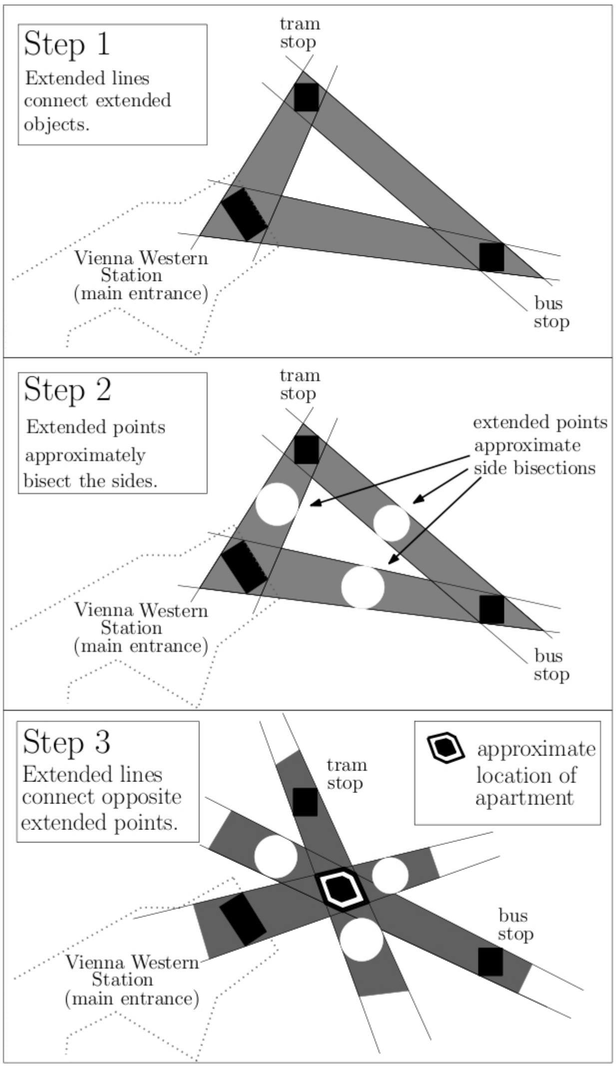

Figure 3. Ad-hoc example of a geometric construction based on the extended geometric primitives "extended point" and "extended line".

1.2 Related Work

Most of the literature on qualitative spatial reasoning in the context of GIS is either topological or metrical in nature (Freksa, 1991; Frank, 1992; Dilo, 2006; Renz and Nebel,2007). Many of these approaches use fuzzy set theory to represent uncertain or incomplete information. The reasoning mechanism itself usually employs crisp calculi. It is rarely the case that fuzzy logic is utilised as a reasoning technique.

One of the approaches that use fuzzy theory for spatial reasoning has been introduced by S. Dutta (1990) for geometric and metric concepts. Dutta uses fuzzy approximate reasoning to propagate positional, metrical, propositional, and range constraints through the steps of a geometric construction process. His approach is conceptually similar to the present work, but does not develop a systematic approximate calculus based on axiomatic geometry. H. Schmidtke (2005) provides an axiomatic geometric approach to spatial reasoning, but focuses more on granularity issues than on geometric constructions. S. Schockaert (Schockaert et al., 2008) employs fuzzy reasoning techniques to define metrical relations like *near* and *far* between extended geographic entities, but does not address geometric constructions. E. Clementini (2005) proposes a geometric model for uncertain lines, but does not treat lines as geometric primitives.

There are numerous approaches by mathematicians to restore Euclidean Geometry from a different set of axioms, based on primitives that have extension in space: (Tarski, 1956) developed a G*eometry* O*f Solids* based on the notions of sphere and inclusion between spheres. (Schmidt, 1979) starts off with regions, an inclusion relation for regions, translations and rotations. The primitives in Gerla's *Point-Free Geometry* (Gerla, 1990; Gerla, 1995) are regions. Extensionless points are defined by a suitable sequence of regions, called *abstraction process*. Bennett (Bennett et. al., 2000; Bennett, 2001) continues on Tarski's Geometry of Solids with *Region Based Geometry*. Region Based Geometry is based on a congruency relation and is formalized exclusively in first order logic. These approaches aim at restoring Euclidean geometry, including the concepts of crisp points and lines, starting from different primitive objects and relations. In contrast to this, the present approach aims at augmenting an existent axiomatization of Euclidean geometry with grades of validity for axioms. The concept of a graded validity of axioms admits models of partial truth, allowing for primitives that have uncertainty in location. A parallel calculus has the advantage of enabling GIS users to use the classical tools of spatial analysis without learning new and fundamentally different concepts.

2. AXIOMATIC GEOMETRY AND EXTENDED OBJECTS

2.1 Geometric primitives and incidence

Euclidean geometry in its axiomatic form was introduced by Euclid in 300BC in his famous book *Elements*. In 1899 David Hilbert gave a complete and consistent formulation of an axiomatic system of Euclidean geometry (Hilbert 1962). The primitive objects in the two dimensional version of his formulation are *points* and *lines*. The most basic primitive relation between points and lines is the *on*-relation, usually called *incidence*. The following four axioms formalize the behaviour of points and lines with respect to incidence:

(I1) For every two distinct points p and q, at least one line l exists that is incident with p and q.
(I2) Such a line is unique.
(I3) Every line is incident with at least two points.
(I4) At least three points exist that are not incident with the same line.

Whenever a set of objects called *points*, another set of objects called *lines*, and a relation called *incidence* comply with these four axioms, the structure is called a (model of) *incidence geometry*. Points, lines and incidence are called *primitives* of the theory. The underlying predicate logic provides a deduction system, which allows deriving theorems from the axioms I1-I4.

An example of a theorem of incidence geometry is the statement "For every two distinct lines at most one point exists, such that both lines are incident with that point." In other words, two distinct lines are either parallel or intersect in exactly one point.

The uniqueness axiom I2 ensures that geometrical constructions are possible. Geometric constructions are sequential applications of construction operators. An example of a construction operator is $connect: \text{point} \times \text{point} \rightarrow \text{line}$, taking two points as an input and returning the line through them. For *connect* to be a well defined mathematical function, the resulting line needs always to exist and needs to be unique. Other examples of geometric construction operators of 2D incidence geometry are

$$\begin{aligned} &intersect: \text{line} \times \text{line} \rightarrow \text{point}, \\ &parallel\ through\ point: \text{line} \times \text{point} \rightarrow \text{line}. \end{aligned} \qquad (1)$$

For the successful implementation of geometric algorithms in GIS, like for example a point-in-polygon-test, the construction of a Voronoi-diagram, or polygon-overlay, the existence of well-defined constructions operators is obligatory.

The axioms of incidence geometry form a proper subset of the axioms of Euclidean geometry. Incidence geometry allows for defining the notion of parallelism of two lines as a derived concept, but does not permit to express betweenness or congruency relations, which are assumed primitives in Hilbert's system. The complete axiom set of Euclidean geometry provides a greater number of construction operators than incidence geometry. Incidence geometry has very limited expressive power when compared with the full axiom system. Due to its small number of axioms incidence geometry is well suited for demonstrating the proposed framework.

The following subchapter focuses on the discussion of a well-defined *connect* operator for extended objects. We give five examples of possible interpretations of the geometric primitives point, line, and incidence by extended objects, test them for compliance with the axioms I1-I4, and discuss their usefulness in a GIS-context.

2.2 Connecting extended points

The combined incidence axioms I1 and I2 state that it is always possible to connect two distinct points by a unique line. In case of coordinate points p and q, Cartesian geometry provides a formula for constructing this unique line: The parametric form reads

$$l = \{ p + t(q-p) \mid t \in \mathbb{R} \}. \qquad (2)$$

When we want to connect two extended geographic objects in a similar way, there is no canonical way of doing so. We can not refer to an existing model like the Cartesian algebra. Instead, a new way of interpreting geometric primitives must be found, such that the interpretation of the incidence relation respects the uniqueness property I2. In the following we will show that such an interpretation cannot be found without imposing too restricting conditions on the interpretation of extended primitives to be useful in a GIS context.

We will refer to extended objects that play the geometric role of points and lines by *extended points* and *extended lines*, respectively. In contrast, the extensionless coordinate points and lines of Cartesian geometry will be denoted by *Cartesian points* and *Cartesian lines*.

Interpretation 1 (Figure 4a): As a first attempt to find an interpretation of I1-I4 with extended primitives, we interpret *extended points* as discs in the Cartesian plane $\mathbb{R}^2$ with a fixed diameter ϕ. Extended lines are read as parallel stripes $L \subset \mathbb{R}^2$ of fixed width ϕ. A stripe *L* is taken to be incident with a point *P*, if $P \cap L = P$. With these definitions, a unique connection operation is defined: It is clear that, for any two distinct points *P* and *Q*, a parallel stripe *L* of width ϕ exists that is incident with both, *P* and *Q.* Such a stripe is unique, and thus axioms 1 and 2 hold. The third and fourth axioms hold trivially, as long as the workspace is big enough. This interpretation is isomorphic to the Cartesian model. It's applicability to reasoning with extended geographic objects in a GIS is limited, since it can not handle objects of different size and shape.

Interpretation 2 (Figures 4b, 4c): In the case that P and Q are disc-shaped, but are allowed to have varying diameters $\phi_P, \phi_Q \in \mathbb{R}$ the above interpretation of connection loses the uniqueness property (Figures 4b, 4c).

Figures 4d-4f sketch two possibilities to restore uniqueness by changing the interpretations of *extended lines* and the *incidence*. In all cases, there seems to be a trade-off between uniqueness and usefulness for GIS purposes:

Interpretation 3 (Figure 4d): Extended lines $L_3 = (P, Q)$ are interpreted as the Cartesian convex hull of pairs of extended points *P* and *Q*. An extended point *R* is taken to be incident with $L_3 = (P, Q)$, if *R*=*P* or *R*=*Q*. As a result, the connection operation exists and is unique. Yet, every extended line so defined contains two points at maximum, which is not an intended understanding of "extended line" for GIS purposes. E.g. the continuation of L_3 to the left of *P* and to the right of *Q* is not defined.

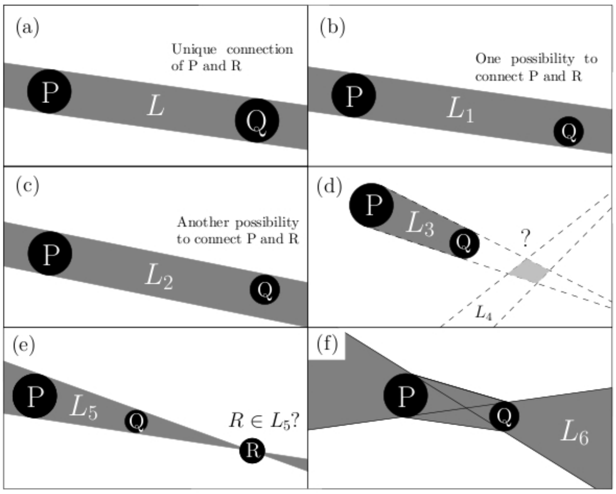

Figure 4. Different interpretations of the connection of two extended points.

Interpretations 4 and 5 (Figures 4e, 4f) : Figures 4e and 4f propose two possibilities of continuation of the convex hull. Both variants impose additional constraints on extended lines that are not derived from the data. These artificially added constraints create new constraints on subsequently constructed objects. For instance, the extended point *R* in Figure 4e is a translation of *Q* in the "main direction" of L_5. Intuitively, *R* should be incident with L_5, which it is not. Only if we shrink *R* to a Cartesian point, the incidence relation is satisfied.

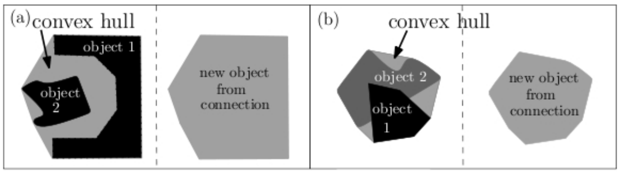

Figure 5. Convex-hull interpretation of the connection of two extended points (a) for arbitrary shapes, (b) for overlapping Cartesian point sets.

In case we additionally allow arbitrary shapes and drop the condition that extended points must not overlap, the different interpretations of connection can become even less useful: Figure 5 shows two constellations where the connection of P and Q by interpretation 3 seems to result in a new extended point rather than in an object that represents an extended linear feature.

The above considerations suggest that an interpretation that is based on extended primitives and complies with the axioms I1-I4 cannot be found, if we demand a definition of extended primitives that is flexible enough to be useful for GIS purposes. Yet, since interpretation 1 is based on extended primitives and complies with the incidence axioms, we conclude that the difficulties we encountered above do not arise from the absolute sizes of extended geometric objects involved or from the absolute distances between them. Instead, problems seem to stem from differences in size and distance of the involved objects relative to each other.

2.3 Approximating incidence geometry

To escape the dilemma encountered in the forgoing subchapter we propose to fuzzify the Cartesian model of incidence geometry. This can be done in three steps: First, we interpret the geometric primitives *point*, *line* and *incidence* as logical predicates and fuzzify their Cartesian interpretation. Secondly we fuzzify the background language of predicate logic, in which the incidence axioms are expressed. And thirdly the associated deduction system itself is fuzzified.

For the first step, to define a fuzzification of a Cartesian point w.r. to its geometric characteristics, we start from the observation that an extended object *P* always comprises a set of Cartesian points, and consequently may be seen as a set of possible or actual locations of a single Cartesian point as permitted by the location constraint *P*. Baring this viewpoint in mind, we may interpret both, extended points and extended lines by arbitrary Cartesian subsets of the real plane $\mathbb{R}^2$, and assign to each of them a degree which expresses how much they "resemble" a Cartesian point or a Cartesian line w.r. to geometric constructions. In this understanding, every Cartesian point set is at the same time a - more or less good - approximation of an extensionless Cartesian point and a - more or less good - approximation of an extensionless Cartesian line.

For the second step, the fuzzification of Boolean predicate logic, note that Boolean predicate logic assumes that predicates can assume either the truth value true ("1"), or the truth value false ("0"). To fuzzify Boolean predicate logic we use infinite valued Łukasiewicz predicate logic, which allows for truth values in the interval [0,1].

Despite the fact that Łukasiewicz logic allows for fuzzy predicates assuming truth values in [0,1], but its deduction system only propagates absolute truth. To implement the third

step, Rational Pavelka Logic (RPL) is proposed. RPL provides an extension of Łukasiewicz logic that allows for deducing partially true conclusions from partially true premises (Hajek, 1998). In this sense, it is a fuzzification of the deduction apparatus of Boolean predicate logic.
The following chapter 3 gives a brief introduction in fuzzy logic and discusses possible interpretations of fuzzy predicates for extended geometric primitives. Based on these primitives an fuzzification of the incidence axioms I1-I4 is proposed and Rational Pavelka logic is introduced as a possible formalism for approximate geometric reasoning with extended objects.

3. FUZZIFICATION OF INCIDENCE GEOMETRY

3.1 Fuzzy logic

Fuzzy logic is derived from fuzzy set theory, which was introduced 1965 in the seminal paper (Zadeh, 1961) by Lotfi Zadeh. In a narrow sense, fuzzy logic is a form of multi-valued logic: Łukasiewicz fuzzy logic was originally defined as early as 1917 by Jan Łukasiewicz as a three valued propositional calculus. It was the first axiomatization of a non-classical logical system. In contrast to that, infinite valued Łukasiewicz fuzzy predicate logic is a multi-valued predicate logic that allows for not only three truth values, but for truth values in the whole range of real numbers of the interval [0, 1]. It belongs to the class of t-norm fuzzy logics: a t-norm is a generalization of the AND connective of classical Boolean logics and can be used to define other logical connectives in an appropriate way.
In Łukasiewicz predicate logic the connectives *negation* $\neg$, *strong conjunction* $\otimes$, and *implication* $\rightarrow$ are evaluated by

$$\neg x = 1 - x \ , \quad (3)$$

$$x \otimes y = \max\left\{0, x + y - 1\right\} \text{ , and} \quad (4)$$

$$x \rightarrow y = \min\left\{1, 1 - x + y\right\} \ , \quad (5)$$

for $x, y \in [0,1]$. The quantifiers *for all* $\forall$ and *exists* $\exists$ are evaluated by the infimum *inf* and the supremum *sup*, respectively. For the implication $\rightarrow$ the following relation holds:

$$x \rightarrow y = 1 \ \Leftrightarrow \ x \leq y \ . \quad (6)$$

The narrow understanding of fuzzy logic, indicating different forms of multi-valued logical systems, is contrasted by *fuzzy logic in the broader sense*. In the latter understanding, fuzzy logic comprises diverse tools for approximate reasoning (Zadeh, 1975). Rational Pavelka Logic provides a strictly logical formal deduction system. Yet, within the system, a syntactically derived truth value of a formula can be less than the "real" truth value of the formula, which is defined by semantic entailment. So we may interpret the syntactically derived truth value as information on a worst case scenario for the given formula.
Once an RPL-model of Euclidean geometry is found, the deduction system provides a computationally inexpensive extension of Cartesian geometry: Every formula is augmented by a rational number indicating the formula's worst case truth value. In the spirit of approximate reasoning and fuzzy logic in the broader sense, accurate, but often too complex information on the well-definedness of a geometric formula is traded against an approximate, but slim calculus, which can be easily implemented by augmenting existing algorithms for Cartesian geometry.
In the next subchapter we propose a fuzzy interpretation of the geometric primitives *point*, *line*, *incidence* and *equality*. Since the geometric behaviour of extended objects depend on the relative sizes and distances of the involved objects, an additional predicate is introduced, which tries to capture this fact: In addition to the possible negation of equality of objects, a measure for the *distinctness of points* is given.

3.2 Geometric primitives as fuzzy predicates

In Boolean predicate logic atomic statements are formalized by predicates. Predicates that are used in the theory of incidence geometry may be denoted by *p(x)* ("x is a point"), *l(x)* ("x is a line"), and *inc(x,y)* ("x and y are incident"). The predicate expressing equality can be denotes by *eq(x,y)* ("x and y are equal"). Predicates are interpreted by crisp relations. For example, $eq : M \times M \rightarrow \{0,1\}$ is a function that assigns 1 to every pair of equal objects and 0 to every pair of distinct objects from the set *M*. Predicates have an *arity*: *unary* predicates, like *p(.)* or *l(.)*, accept only one symbol as an input, whereas *binary* predicates, like *inc(.,.)* and *eq(.,.)*, accept pairs of symbols as an input.
In a fuzzy predicate logic, predicates are interpreted by fuzzy relations, instead of crisp relations. For example, a binary fuzzy relation *eq* is a function $eq : M \times M \rightarrow [0,1]$, assigning a real number $\lambda \in [0,1]$ to *every* pair of objects from *M*. In other words, every two objects of *M* are equal to some degree. The degree of equality of two objects *x* and *y* may be 1 or 0 as in the crisp case, but may as well be 0.9, expressing that x and y are *almost* equal.
In the following we propose a possibility to fuzzify the Boolean predicates *point(.)*, *line(.)* *inc(.,.)* and *eq(.,.)* for GIS. We define a bounded subset $D \subseteq \mathbb{R}^2$ as the domain for our geometric constructions. We may restrict ourselves to a bounded domain, because every GIS project has a bounded domain $Dom \subset \mathbb{R}^2$: *Dom* represents the map or map section we are working with. Predicates are defined for two-dimensional subsets *A,B,C,…* of *Dom*, and assume values in *[0,1]*. We may assume two-dimensional subsets and ignore subsets of lower dimension, because every measurement and every digitization introduces a minimum amount of location uncertainty in the data (Goodchild, 2000).
For the point-predicate *p(.)*, we start from the observation that the result of Cartesian geometric operations that involve a Cartesian point does not change when the point is rotated: Rotation-invariance seems to be a main characteristic of "pointlikeness" w.r. to geometric operations: It should be kept when defining a fuzzy predicate expressing the "pointlikeness" of extended subsets of $\mathbb{R}^2$. As a preliminary definition let

$$\phi_{\min}(A) = \min_t \left| ch(A) \cap \left\{ c(A) + t \cdot R_\alpha \cdot (0,1)^T \mid t \in \mathbb{R} \right\} \right| , \quad (7)$$

$$\phi_{\max}(A) = \max_t \left| ch(A) \cap \left\{ c(A) + t \cdot R_\alpha \cdot (0,1)^T \mid t \in \mathbb{R} \right\} \right| , \quad (8)$$

be the minimal and maximal diameter of the convex hull *ch(A)* of $A \subseteq Dom$, respectively. The convex hull regularizes the sets A and B and eliminates irregularities. *c(A)* denotes the centroid of *ch(A)*, and R_α denotes the rotation matrix by angle α (Figure 6a). Since *A* is bounded, *ch(A)* and *c(A)* exist. We can now define the fuzzy point-predicate *p(.)* by

$$p(A) = \frac{\phi_{\min}(A)}{\phi_{\max}(A)} \quad (9)$$

for $A \subseteq Dom$. *p(.)* expresses the degree to which the convex hull of a Cartesian point set *A* is rotation-invariant: If *pl(A)=1*, then *ch(A)* is perfectly rotation invariant; it is a disc. Here, $\phi_{\max}(A) \neq 0$ always holds, because *A* is assumed to be two-dimensional.
Converse to *p(.),* the fuzzy line-predicate

$$l(A) = 1 - pl(A) \quad (10)$$

expresses the degree to which a Cartesian point set $A \subseteq Dom$ is sensitive to rotation. Since we only regard convex hulls, *l(.)* disregards the detailed shape and structure of *A*, but only measures the degree to which *A* is directed.
A fuzzy version of the incidence-predicate *inc(.,.)* is a a binary fuzzy relation between Cartesian point sets $A, B \subseteq Dom$:

$$inc(A,B) = \max\left(\frac{|ch(A) \cap ch(B)|}{|ch(A)|}, \frac{|ch(A) \cap ch(B)|}{|ch(B)|}\right) \quad (11)$$

measures the relative overlaps of the convex hulls of *A* and *B* and selects the greater one. Here *|ch(A)|* denotes the area occupied by *ch(A)*. The greater *inc(A,B)*, "the more incident" are *A* and *B*: If $A \subseteq B$ or $B \subseteq A$, then *inc(A,B)=1*, and *A* and *B* are considered *incident to degree one*.
Conversely to *inc(.,.)*, a graduated equality predicate *eq(.,.)* between the bounded Cartesian point sets $A, B \subseteq Dom$ can be defined as follows:

$$eq(A,B) = \min\left(\frac{|ch(A) \cap ch(B)|}{|ch(A)|}, \frac{|ch(A) \cap ch(B)|}{|ch(B)|}\right). \quad (12)$$

$eq(A,B)$ measures the minimal relative overlap of *A* and *B*, whereas $\neg eq(A,B) = 1 - eq(A,B)$ measures the degrees to which the two point sets do not overlap: if $eq(A,B) \approx 0$, then *A* and *B* are "almost disjoint".
When defining *p(A)* and *l(A)* for a bounded Cartesian point set *A*, it is not necessary to take the absolute size of *A* into account. As stated at the end of chapter 2.2, only relative sizes and distances seem to cause ill-posed geometric constellations. The following measure of "distinctness of points", *dp(.,.)*, of two extended objects tries to capture this fact (Figure 6b). We define

$$dp(A,B) = \max\left(0, 1 - \frac{\max\left(\phi_{\max}(A), \phi_{\max}(B)\right)}{\phi_{\max}\left(ch(A \cup B)\right)}\right). \quad (13)$$

dp(A,B) expresses the degree to which *ch(A)* and *ch(B)* are distinct: The greater *dp(A,B)*, the more *A* and *B* behave like distinct Cartesian points w.r.t. connection. Indeed, for Cartesian points *a* and *b*, we would have *dp(a,b)=1*. If the distance between the Cartesian point sets *A* and *B* is infinitely big, then *dp(A,B)=1* as well. If $max\left(\phi_{\max}(A), \phi_{\max}(B)\right) > \phi_{\max}\left(A \cup B\right)$, then *dp(A,B)=0*.

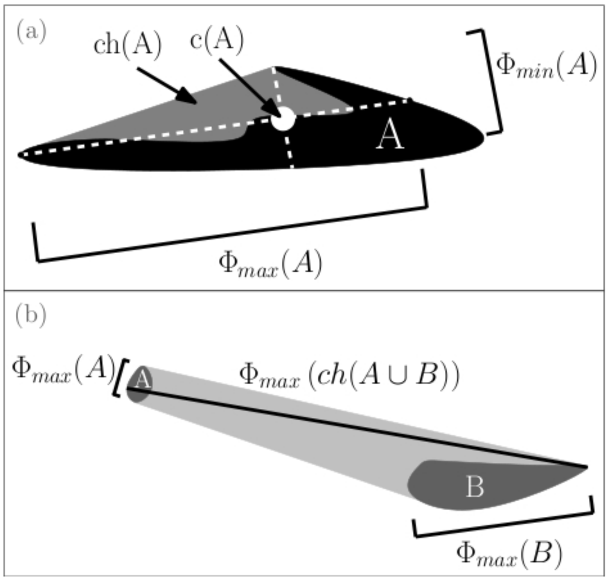

Figure 6. (a) Minimal and maximal diameter of a set *A* of Cartesian points. (b) Grade of distinctness *dc(A,B)* of *A* and *B*.

Example: The polygon representing the entrance of Vienna Western Station (*E*) in Figure 1 and the polygon representing the tram stop (*T*) have a degree of distinctness of points of *dp(E,T)*=0.8. In contrast to that, the polygon comprising of the whole rail yard (*R*) of Vienna Western Station in Figure 2 and *T* have a degree of distinctness of points of *dp(R,T)*=0. The value of the line-predicate of $ch(E \cup T)$ and $ch(R \cup T)$ amounts to $l\left(ch(R \cup T)\right) = 0.9$ and $l\left(ch(R \cup T\right) = 0.3$, respectively.

3.3 Fuzzy axiomatization of incidence geometry

Using the fuzzy predicates defined in subchapter 3.2, we axiomatize a fuzzy version of incidence geometry in the language of Łukasiewicz logic as follows:

$$\text{I1}' \quad dp(x,y) \rightarrow \sup_z \left[l(z) \otimes inc(x,z) \otimes inc(y,z) \right]$$

$$\text{I2}' \quad dp(x,y) \rightarrow \left[l(z) \rightarrow \left[inc(x,z) \rightarrow \left[inc(y,z) \rightarrow l(z') \rightarrow \left[inc(x,z') \rightarrow \left[inc(y,z') \rightarrow eq(z,z') \right] \right] \right] \right] \right]$$

$$\text{I3}' \quad l(z) \rightarrow \sup_{x,y} \left\{ p(x) \otimes p(y) \otimes \neg eq(x,y) \otimes inc(x,z) \otimes inc(y,z) \right\}$$

$$\text{I4}' \quad \sup_{u,v,w,z} \left[p(u) \otimes p(v) \otimes p(w) \otimes l(z) \rightarrow \neg\left(inc(u,z) \otimes inc(v,z) \otimes inc(w,z) \right) \right]$$

An interpretation of the fuzzy predicates *p(.)*, *l(.)*, *inc(.,.)*, *eq(.,.)*, and *dp(.,.)* is called a *model* of I1′-I4′, if each axiom evaluates with truth value 1, independently of the substitution of specific Cartesian point sets for *x,y,z,u,v,w*. Furthermore, the equality predicate *eq(.,.)* should evaluate to truth value 1 for each of the fuzzified equality-axioms - reflexivity, symmetry and transitivity - of predicate logic. This is not the case: For example, *eq(.,.)* violates the transitivity condition. To see this, consider the Cartesian point sets *A, B, C* as sketched in figure 7. On the one hand *eq(A,B)=eq(B,C)=0.75*, and *eq(A,C)=0* holds for *A,B,C*. On the other hand, the transitivity axiom for *eq(.,.)* demands that

$$eq(A,B) \otimes eq(B,C) \rightarrow eq(A,C) \quad (14)$$

holds with truth value 1, i.e. that

$$eq(A,B)\otimes eq(B,C)\rightarrow eq(A,C) \quad =1\,. \qquad (15)$$

With (6), (15) is equivalent to

$$eq(A,B)\otimes eq(B,C)\leq eq(A,C)\,. \qquad (16)$$

Yet, (4) yields $eq(A,B)\otimes eq(B,C)=0.5$, which contradicts (16).

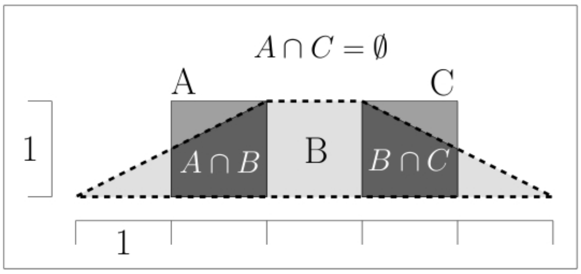

Figure 7. The squares A and C, together with the trapezoid B, refute the transitivity of the *eq(.,.)* predicate (11).

In the next chapter, Rational Pavelka Logic (RPL) is discussed. RPL allows for deducing new formulas from I1´-I4´, even if the axioms do not evaluate to absolute truth for all possible inputs.

3.4 Rational Pavelka Logic

Rational Pavelka Logic (RPL) extends the language of infinite valued Łukasiewicz logic by adding to the truth constants 0 and 1 all rational numbers *r* of the unit interval [0, 1]. A *graded formula* is a pair (φ,r) consisting of a formula φ of Łukasiewicz logic and a rational element $r\in[0,1]$, indicating that the truth value of φ is at least *r*, $\varphi\geq r$. For example, *(p(x), ½)* expresses the fact that the truth value of *p(x),* $x\subseteq Dom$, is at least ½. In other words, *x* resembles a point at least with degree 0.5.

The inference rules of RPL are the *generalization rule*

$$\frac{\varphi}{(\forall x)(\varphi)}\,, \qquad (17)$$

and a modified version of the *modus ponens rule*,

$$\frac{(\varphi,r),(\varphi\rightarrow\psi,s)}{(\psi,r\otimes s)}\,, \qquad (18)$$

where $\otimes$ denotes the Łukasiewicz t-norm. Rule (18) says that if formula φ holds at least with truth value *r*, and the implication $\varphi\rightarrow\psi$ holds at least with truth value *s*, then formula ψ holds at least with truth value $r\otimes s$. The modified modus ponens rule (15) is derived from the so-called *book-keeping axioms* for the rational truth constants *r*. The book-keeping axioms add to the axioms of Łukasiewicz logic and provide rules for evaluating compound formulas involving rational truth constants (Hajek, 1998).

In RPL, we axiomatize a fuzzy version of incidence geometry as follows:

I1´´ $\left(dp(x,y)\rightarrow\sup_z\left[l(z)\otimes inc(x,z)\otimes inc(y,z)\right],\ r_1\right)$

I2´´ $\big(dp(x,y)\rightarrow\big[l(z)\rightarrow\big[inc(x,z)\rightarrow\big[inc(y,z)\rightarrow$
$l(z')\rightarrow\big[inc(x,z')\rightarrow\big[inc(y,z')\rightarrow eq(z,z')\big]\big]\big]\big],\ r_2\big)$

I3´´ $\left(l(z)\rightarrow\sup_{x,y}\left\{p(x)\otimes p(y)\otimes\neg eq(x,y)\otimes inc(x,z)\otimes inc(y,z)\right\},\ r_3\right)$

I4´´ $\Big(\sup_{u,v,w,z}\ \big[p(u)\otimes p(v)\otimes p(w)\otimes l(z)\rightarrow$
$\neg\big(inc(u,z)\otimes inc(v,z)\otimes inc(w,z)\big)\big],r_4\Big)$,

where r_1,r_2,r_3,r_4 are rational truth constants.

An interpretation of the predicates *p(.)*, *l(.)*, *inc(.,.)*, *eq(.,.)*, and *dp(.,.)* is a *model* of I1´´-I4´´, if, for each of the graded axioms (α,r_α) , $\alpha\geq r_\alpha$ holds independently of the substitution of specific Cartesian point sets for *x,y,z,u,v,w*.

A syntactically derived formula is a graded formula, that has been derived from the axioms of RPL and the axiom set I1´-I4´ by use of the inference rules (17) and (18). Yet, using this deduction apparatus, the same formula may be derived in different ways and with different truth values attached. For this reason a provability degree for formulas is defined: The *provability degree* of a formula φ is the highest truth value that can be syntactically derived for φ . In contrast to that, the *truth degree* of φ is the lowest truth value that is semantically implied by the axioms. It is the semantic equivalent to the provability degree. The truth degree of a formula can be seen as the “real” truth value of the formula.

The provability degree of a formula is always less or equal than its truth degree. Consequently, for every formula that is syntactically derived by the RPL deduction system, the “real” truth value is greater or equal than the derived truth value. The derived truth value hence provides a lower bound for the truth of the formula.

If it can be shown that each of the incidence axioms I1´´-I4´´, together with the interpretation of fuzzy predicates defined in subchapter 2.3, holds for some minimal truth degree of $r_1>0$, …, $r_4>0$, respectively, then I1´´-I4´´ is a fuzzy set of axioms for incidence geometry of extended objects. The inference rules (17) and (18) of RPL can be used to derive partially true theorems from partially true conclusions. Since a derived truth value always is a lower bound for the truth of the derived formula, the derived truth value can be seen as bound for the worst case.

As shown in chapter 3.3, the connection of two extended objects is not necessarily unique for the fuzzy interpretations introduced in chapter 3.2. Depending on the context of the a specific GIS project, it may be useful to select one of these interpretations, e.g. the convex hull, as a fixed, but suboptimal connection operator. Using axiom I2´´, RPL can be used for test runs to find out “how well” the chosen operator performs in comparison with the best possible operator.

4. CONCLUSIONS

4.1 Conclusions

We have shown that straight forward interpretations of the connection of extended points do not satisfy the incidence axioms of Euclidean geometry in a strict sense. Yet, the approximate geometric behaviour of extended objects can be described by fuzzy predicates. Based on these predicates, the axiom system of Boolean Euclidean geometry can be fuzzified and formalized in the language of Lukasiewicz fuzzy logic.

As an approximate deduction system, Rational Pavelka Logic is proposed. Rational Pavelka Logic derives partially true

conclusions from partially true premises and thereby provides lower bounds for the truth values of geometric formulas. This allows for tolerance in the truth value of geometric formulas w. r. to the extended objects that serve as input to the formla in question. As a consequence, the derived truth values allow for the possibility to warn users, in case a geometric constellation of extended objects is not sufficiently well-posed for a specific operation.
The use of fuzzy reasoning trades accuracy against speed, simplicity and interpretability for lay users. In the context of ubiquitous computing, these characteristics are clearly advantageous.

4.2 Discussion and further work

The axiom set I1´-I4´ is an ad-hoc fuzzification of the axioms of incidence I1-I4. The predicates *p(.), l(.), inc(.,.), eq(.,.)* and *dp(.,.)* do not satisfy the fuzzified incidence axioms I1´-I4´ to degree 1. A detailed analysis of the interaction between the interpretation of the predicates and the axiomatization is necessary.
The axiom system I1´´-I4´´, together with the fuzzy primitives RPL-fuzzification of incidence axioms *p(.), l(.), inc(.,.), eq(.,.)* and *dp(.,.)* is a reasonable suggestion for an approximate geometric calculus of extended primitives. The proof of existence of positive constants $r_1 > 0$, …, $r_4 > 0$ is obligatory for an implementation of the proposed framework and is left for future work.
The set of incidence axioms discussed in the present article is only one out of five axiom groups of Hilbert's axiomatic system of Euclidean geometry. In further work, we will extend the set of fuzzy predicates to *betweenness* and *congruence* and the according axiom groups will be fuzzified. Due to the boundedness of the domain *Dom*, the axiom group dealing with *continuity* will be omitted.

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