APPROACH ON AREA COORDINATE, VOLUME COORDINATE AND THEIR USAGE IN TRUE 3DGIS

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ABSTRACT:

Area coordinate and the volume coordinate are local coordinate which have been used in the computer geometry modelling and the engineering mechanics finite element calculation. The area coordinate is calculated by ratio of triangle area with the polygon area, the triangle is enclosed by moving point and a boundary edge. The volume coordinate is calculated by ratio of sub-polyhedron volume with the mother polyhedron volume, the sub-polyhedron is enclosed by moving point and a boundary face. The common area coordinates are the triangle area coordinate and the quadrilateral area coordinate. The common volume coordinate are the tetrahedron volume coordinate and hexahedron volume coordinate. The area/volume coordinate possess many excellent characters, which the rectangle coordinate and polar coordinate do not possess. This paper introduces the basic theory about these local coordinate. Especially, the paper detail gives the author's research work about the 3D triangle area coordinate and tetrahedron volume coordinate. The author summarized the eight characters of the tetrahedron volume coordinate. Then the paper gives usage design about the area /volume coordinate application in true 3DGIS.

1. INTRODUCTION

The engineering mechanics academy said that "for engineering mechanics, to find out the area coordinate is as if one finds out treasure". By many application domains pushed, the area coordinate has been developed from triangular area coordinate to quadrangular area coordinate, even to volume coordinate in 3D world. Area coordinate and volume coordinate have become a new scientific growth point for many application domains.

Firstly, this paper describes basic theory of area coordinate and volume coordinate. Then the paper presents author's research work about three-dimensional triangular area coordinate and tetrahedron volume coordinate. Thirdly, some conceive plans about use the area/volume coordinate in true three-dimensional GIS are given. The authors hope to introduce the area/volume coordinate theory to GIS science.

2. AREA COORDINATE THEORY

2.1 Area Coordinate concept

The area coordinate is a kind of 2D local coordinate, which is composed by a sequence of ratios of small triangles with a polygon, the small triangle is surround by a moving point and one edge of the polygon. The common area coordinate are triangular area coordinate and quadrangular area coordinate. The triangular area coordinate was studied firstly by mathematics. In recent years, it has been paid attention by the computer geometric modelling to express nature surface, and by engineering mechanics to expression stress distribution in the finite element method. By the engineering mechanics application pushed, the area coordinate has been developed from triangular area coordinate to quadrangular area coordinate.

2.2 Triangle Area Coordinate concept

The triangle area coordinate is also called triangle barycentre coordinate. The triangle area coordinate is formed by a moving point P(x ,y) in a triangle △ABC. The moving point P with three edges BC, CA, AB form three small triangles (△BCP, △CAP, △ABP). The sequence of the three ratio values (u, v, w) of area of the three triangle with the mother triangle △ABC area is a triangle area coordinate.

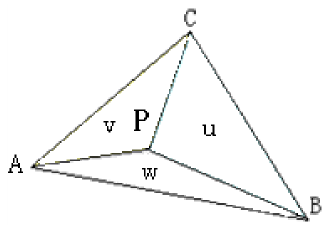

Figure 1. Triangle area coordinate

$$u = \Delta BCP/\Delta ABC \quad v = \Delta CAP/\Delta ABC \quad w = \Delta ABP/\Delta ABC \qquad (1)$$

Here, It is defined that if the vectors are anti-clockwise linked, the area is positive, else clockwise linked, the area is negative. Constant have u + v + w = 1.0, that is, u, v, w three variables, only two are independent.

With $A(X_1,Y_1)$,$B(X_2,Y_2)$,$C(X_3,Y_3)$ rectangular coordinates:

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$$u = [(X_2Y_3 - X_3Y_2) + (Y_2 - Y_3)x + (X_3 - X_2)\, y\,] / (2S)$$
$$v = [(X_3Y_1 - X_1Y_3) + (Y_3 - Y_1)x + (X_1 - X_3)\, y\,] / (2S) \qquad (2)$$
$$w = [(X_1Y_2 - X_2Y_1) + (Y_1 - Y_2)x + (X_2 - X_1)\, y\,] / (2S)$$

It also can be written as:

$$u = [\, x\,(Y_2 - Y_3) + X_2(Y_3 - y\,) + X_3\,(\,y - Y_2)\,] / (2S)$$
$$v = [X_1(\,y - Y_3) + x(Y_3 - Y_1) + X_3(\,Y_1 - y\,)\,] / (2S) \qquad (3)$$
$$w = [X_1(Y_2 - y) + X_2(Y_1 - y\,) + x\,(\,Y_1 - Y_2)\,] / (2S)$$

Here, $S = [X_1(Y_2 - Y_3) + X_2(Y_1 - Y_3) + X_3\,(Y_1 - Y_2)]/2$ (4)

The conversion between the triangle area coordinates (u, v, w) and the rectangular coordinates (x, y) is:

$$\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad (5)$$

Coordinates of the triangle area has the following properties:

1. In the three coordinates (u, v, w), only two independent coordinates.
2. In the three vertices of A, B, C, the value of triangle area coordinates are (1,0,0), (0,1,0), (0,0,1).
3. In the AB edge, when the fixed point changes from A to B, the coordinates from (1,0,0) by (u, v, 0) changes to (0,1,0); Similarly, in BC edge, the coordinates from (0,1,0) by (0, v, w) changes to (0,0,1); edge in CA coordinates from (0,0,1) through (u, 0, w) change to (u, 0, w).
4. Straight line AB, BC, CA respectively w, u, v values greater than 0 and less than 0 line. By determining u, v, w value of the moving point to know which area in the plane. When u> 0, v> 0, w> 0, fixed point within the triangle, u <0, fixed point in BC outside the half-plane, v <0, fixed point in the AC outside, w <0 fixed point other than in BC.

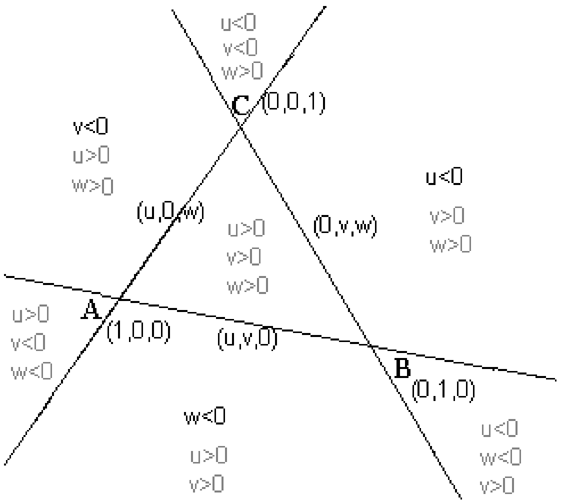


Figure 2. By values of u, v and w, it can be fixed on which area is the moving point located

Due to triangle area coordinate with a lot of beautiful natures, and easy to convert with the rectangular coordinates in three-dimensional computer modelling is widely used in the field of small triangular patches to build free-form surfaces. Detail in the content see the reference (Jingkou 1994 , Gerald 2001, and Xingxiong 2006).

2.3 Three dimensional triangle area coordinate

The authors have studied the three dimensional spatial triangle coordinate calculation method. In a 3D space, 3D points P_1 (X_1, Y_1, Z_1), P_2 (Z_2 ,Y_2, Z_2), P_3 (X_3 ,Y_3, Z_3) constitute a three-dimensional plane triangle △123. Its area is:

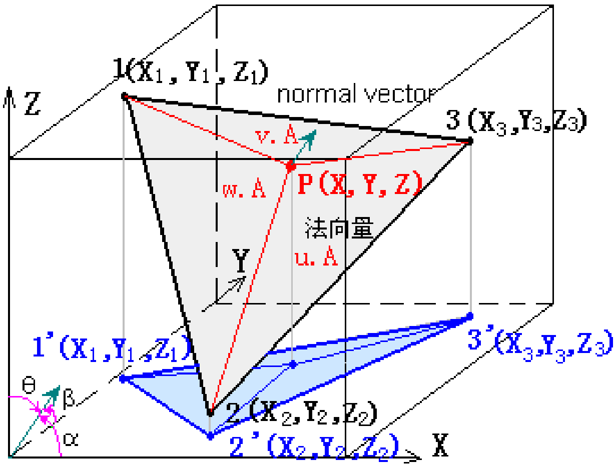


Figure 3. Three dimensional triangle area coordinate

$$\begin{aligned} S=\mathrm{sqrt}\{&[X_1(Y_2-Y_3)+X_2(Y_3-Y_1)+X_3(Y_1-Y_2)\,]^2 \\ &+[Y_1(Z_2-Z_3)+Y_2(Z_3-Z_1)+Y_3(Z_1-Z_2)\,]^2 \\ &+[Z_1(X_2-X_3)+Z_2(X_3-X_1)+Z_3(X_1-X_2)\,]^2\}/2 \end{aligned} \qquad (6)$$

So, the triangular coordinate in 3D space are:

$$\begin{aligned} u=\mathrm{sqrt}\{&[\,x\;(Y_2-Y_3)+X_2(Y_3-y)+X_3(y-Y_2)\,]^2 \\ &+[\,y\;(Z_2-Z_3)+Y_2(Z_3-z)+Y_3(z-Z_2)\,]^2 \\ &+[\,z\;(X_2-X_3)+Z_2(X_3-x)+Z_3(x-X_2)\,]^2\}/(2S) \\ v=\mathrm{sqrt}\{&[X_1(\,y-Y_3)+x\,(Y_3-Y_1)+X_3(Y_1-y)]^2 \\ &+[Y_1(\,z-Z_3)+y\,(Z_3-Z_1)+Y_3(\,Z_1-z)]^2 \\ &+[\,Z_1(\,x-X_3)+z\,(X_3-X_1)+Z_3(X_1-x)]^2\}/(2S) \\ w=\mathrm{sqrt}\{&[\,X_1(Y_2-y)+X_2(\,y-Y_1)+x\,(Y_1-Y_2)]^2 \\ &+[Y_1(Z_2-z)+Y_2(\,z-Z1\,)+y\,(\,Z_1-Z_2)]^2 \\ &+[Z_1(X_2-x\,)+Z_2(\,x-X_1)+z\,(\,X_1-X_2)]^2\}/(2S) \end{aligned} \qquad (7)$$

In most case, the triangle area coordinate is same to take the triangle as 2D or 3D. But, when the triangle plane is near perpendicular, to take 2D coordinate will cause ill-conditioned function. By the author's three dimensional plane triangle calculation method can avoid ill-conditioned surface function, even on the case of the plane is near or whole upstanding.

2.4 Quadrilateral area coordinate

The Quadrilateral area coordinate is developed with limited element method application of engineering mechanics. There are two types of in the quadrilateral area coordinate. This work is come from Long (2007).

2.4.1 Type I quadrilateral area coordinate: The first type quadrilateral area coordinates as shown in Figure 4, for the quadrilateral element 1234, first of all, the definition of the shape of the four dimensionless parameters g1, g2, g3 and g4 are as follows:

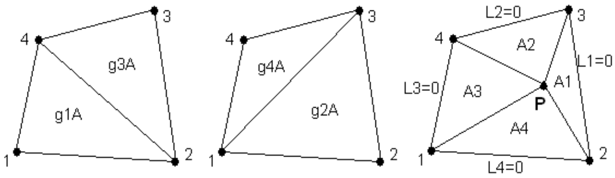


Figure 4. Type I quadrilateral area coordinate

Here,

$$g_1 = g_1A/A; \quad g_2 = g_2A/A; \quad g_3 = 1-g_1, \quad g_4 = 1-g_2 \qquad (8)$$

Of which, A is the area of quadrilateral; g_1A and g_2A are the size of Δ124 and Δ123. It can be seen in only two of four parameters are independent.

As shown in Figure 4, any point inside the quadrilateral element of the quadrilateral area coordinates of the first category (L_1, L_2, L_3 and L_4) is defined as:

$$L_i = A_i/A \quad (i=1,2,3 \text{ and } 4) \qquad (9)$$

In which Ai (i = 1, 2, 3, 4), respectively, while for the P point and 23, edge 34, edge 41 and edge 12 constitutes an area of the triangle.

(L_1, L_2, L_3, L_4) Can also use rectangular coordinates (x, y) is expressed as

$$L_i = (a_i + b_i*x + c_i*y)/(2*A)\ , \quad (i=1,2,3,4) \qquad (10)$$

Or

$$\begin{aligned} L_1 &= (a_1 + b_1*x + c_1*y)/(2A) \\ L_2 &= (a_2 + b_2*x + c_2*y)/(2A) \\ L_3 &= (a_3 + b_3*x + c_3*y)/(2A) \\ L_4 &= (a_4 + b_4*x + c_4*y)/(2A) \end{aligned} \qquad (11)$$

Here,

$$a_i = X_j*Y_k - X_k*Y_j, \quad b_i=Y_j-Y_k, \quad c_i=X_k-X_j, \quad (i=1, 2, 3, 4; j=2, 3, 4, 1; k=3, 4, 1, 2) \qquad (12)$$

That is:

$$\begin{aligned} &a_1=X_2*Y_3 - X_3*Y_2\ , \ b_1=Y_2 - Y_3\ , \ c_1=X_3 - X_2 \\ &a_2=X_3*Y_4 - X_4*Y_3\ , \ b_2=Y_3 - Y_4\ , \ c_2=X_4 - X_3 \\ &a_3=X_4*Y_1 - X_1*Y_4\ , \ b_3=Y_4 - Y_1\ , \ c_3=X_1 - X_4 \\ &a_4=X_1*Y_2 - X_2*Y_1\ , \ b_4=Y_1 - Y_2\ , \ c_4=X_2 - X_1 \end{aligned} \qquad (13)$$

Substitute a_i, b_i, c_i, we can get:

$$\begin{aligned} L_1&=[X_2*Y_3-X_3*Y_2+ (Y_2 - Y_3)*x + (X_3 - X_2)*y)/(2A) \\ L_2&=[X_3*Y_4-X_4*Y_3+ (Y_3 - Y_4)*x + (X_4 - X_3)*y)/(2A) \\ L_3&=[X_4*Y_1-X_1*Y_4+ (Y_4 - Y_1)*x + (X_1 - X_4)*y)/(2A) \\ L_4&=[X_1*Y_2-X_2*Y_1+ (Y_1 - Y_2)*x + (X_2 - X_1)*y)/(2A) \end{aligned} \qquad (14)$$

Or we can write them as(every formula less one multiply)

$$\begin{aligned} L_1 &= [(Y_2 - Y_3)*x + (Y_3 - y)*X_2 + (y - Y_2)*X_3] / (2A) \\ L_2 &= [(Y_3 - Y_4)*x + (Y_4 - y)*X_3 + (y - Y_3)*X_4] / (2A) \\ L_3 &= [(Y_4 - Y_1)*x + (Y_1 - y)*X_4 + (y - Y_4)*X_1] / (2A) \\ L_4 &= [(Y_1 - Y_2)*x + (Y_2 - y)*X_1 + (y - Y_1)*X_2] / (2A) \end{aligned} \qquad (15)$$

Area of quadrilateral 1234 is:

$$A=[X_1(Y_2-Y_4)+X_2(Y_3-Y_1)+X_3(Y_4-Y_2)+X_4(Y_1-Y_3)]/2 \qquad (16)$$

The four vertexes values are:

$P_1(g_4, g_2, 0, 0)$,
$P_2(0, g_3, g_1, 0)$,
$P_3(0, 0, g_4, g_2)$,
$P_4(g_3, 0, 0, g_1)$.

Among them:

$$\begin{aligned} g_1&=\Delta124/A=[X_1(Y_2-Y_4) + X_2(Y_4-Y_1) + X_4(Y_1-Y_2)]/(2A) \\ g_2&=\Delta123/A=[X_1(Y_2-Y_3) + X_2(Y_3-Y_1) + X_3(Y_1-Y_2)]/(2A) \\ g_3&=\Delta234/A=[X_2(Y_3-Y_4) + X_3(Y_4-Y_2) + X_4(Y_2-Y_3)]/(2A) \\ g_4&=\Delta134/A=[X_1(Y_3-Y_4) + X_3(Y_4-Y_1) + X_4(Y_1-Y_3)]/(2A) \end{aligned} \qquad (17)$$

Coordinate values in the four edges are:

In edge 1_2, L_4=0, when point moving from 1 to 2, coordinate change form (g_2,g_4,0,0) to (g_1,g_2,g_3,0) to (0,g_3,g_1,0).

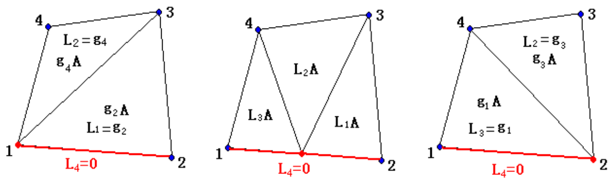


Figure 5. Quadrilateral area coordinate changes on edge 12.

Or:

In vertex 1，$L_1=g_2$, $L_2=g_4$, $L_3=L_4=0$.
In vertex 2，$L_2=g_3$, $L_3=g_1$, $L_1=L_4=0$
L_1: $g_2 \rightarrow 0$, L_2: $g_4 \rightarrow g_3$, L_3: $0 \rightarrow g_3$, $L_4=0$.

The similar coordinate change law also exists in edge 23, 34 and 41.

2.4.2 Type II quadrilateral area coordinate: The second type quadrilateral area coordinate is shown in figure 6. Mi (i = 1, 2, 3, and 4) is middle point in the edge 12, edge 23, edge 34 and edge 41 respectively. So, type II quadrilateral area coordinate is defined as:

$$Z_1 = 4\frac{S_1}{S}, Z_2 = 4\frac{S_2}{S} \qquad (18)$$

Where,

S is still area of quadrilateral element.

S_1 and S_2 are generalized area of ΔPM_2M_4 and ΔPM_3M_1 respectively. They are called as “generalized area” because S_1 and S_2 may be positive or negative. If the sequence order of ΔPM_2M_4 (or ΔPM_3M_1) is anti-clockwise, the S_1(or S_2) area was positive(+), otherwise, S_1 (or S_2) area is negative(－).

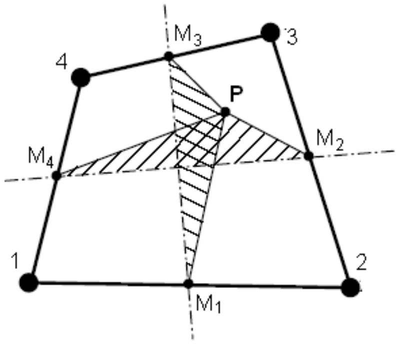


Figure 6 . Type II quadrilateral area coordinate

An advantage of second type quadrilateral area is that they have two variables.

3. VOLUME COORDINATE

3.1 Volume Coordinate concept

The volume coordinate is a expanding of 2D triangle area coordinate into 3D. The volume coordinate is a kind of 3D local coordinate, which is composed by a sequence of ratios of small tetrahedron with the element volume. The small tetrahedron is surrounded by a moving point and one face of the solid element. The common volume coordinate are tetrahedron volume coordinate and hexahedral volume coordinate.

3.2 Tetrahedron volume coordinate

The tetrahedron volume coordinate is a ratio sequence of sub-tetrahedron volumes with the original tetrahedron volumes. A point P moves inside of the original tetrahedron ABCD. P as a vertex with other three vertexes from A, B, C and D forms four sub-tetrahedrons BDCP, ACDP, ADBP and ABCP (Figure.7).

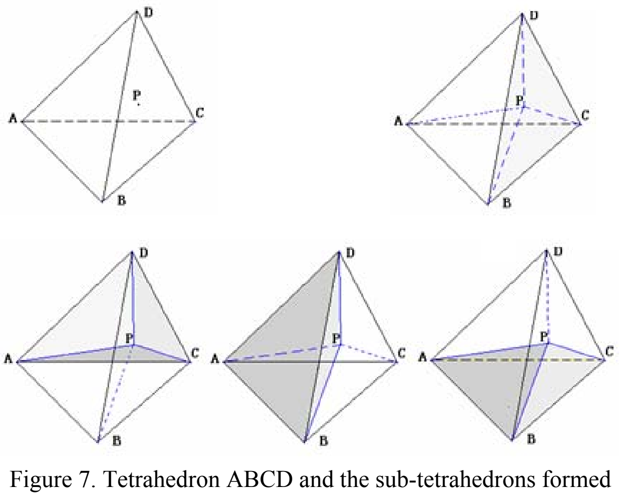


Figure 7. Tetrahedron ABCD and the sub-tetrahedrons formed by moving point P with face

The volume ratios values of sub-tetrahedron with original tetrahedron (u, v, w, r) composed of tetrahedron volume coordinate.

Here, $u = V_{BDC}P / V_{ABCD}$,

$$
\begin{aligned}
u &= V_{BDCP} / V_{ABCD},\\
v &= V_{ACDP} / V_{ABCD},\\
w &= V_{ADBP} / V_{ABCD},\\
r &= V_{ABCP} / V_{ABCD}.
\end{aligned} \tag{19}
$$

$$
V_{ABCD} = \frac{1}{6}(\overrightarrow{AB} \times \overrightarrow{AC}).\overrightarrow{AD} = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ X_A & X_B & X_C & X_D \\ Y_A & Y_B & Y_C & Y_D \\ Z_A & Z_B & Z_C & Z_D \end{vmatrix}
$$

$$
= \frac{1}{6}\begin{vmatrix} X_B - X_A & Y_B - Y_A & Z_B - Z_A \\ X_C - X_A & Y_C - Y_A & Z_C - Z_A \\ X_D - X_A & Y_D - Y_A & Z_D - Z_A \end{vmatrix}
$$

$$
\begin{aligned}
=[\ &(x_D y_C - x_C y_D)(z_A - z_B) + (x_B y_D - x_D y_B)(z_A - z_C) \\
&+ (x_C y_B - x_B y_C)(z_A - z_D) + (x_D y_A - x_A y_D)(z_B - z_C) \\
&+ (x_A y_C - x_C y_A)(z_B - z_D) + (x_B y_A - x_A y_B)(z_C - z_D)\ ]/6,
\end{aligned} \tag{20}
$$

It can be verified that:

$V_{ABCD}=V_{ADBC}=V_{ACDB}=V_{BCAD}=V_{BADC}=V_{BDCA}=V_{CABD}=V_{CDAB}=V_{CBDA}=V_{DACB}=V_{DBAC}$
$=V_{DCBA}=-V_{ACBD}=-V_{ABDC}=-V_{ADCB}=-V_{BACD}=-V_{BDBC}=-V_{BCDA}=-V_{CBAD}$
$=-V_{CADB}=-V_{CDBA}=-V_{DCAB}=-V_{DABC}=-V_{DBCA}$.

Here, the definition of the tetrahedron volume follows the right-handed spiral rule. If the vertices of a tetrahedron following the right hand rule, the volume is for positive (+) and reverse for negative (-), only the exchange of any two vertices constitute a tetrahedral position, the volume value of the symbol upside down.

Based on LIANG's work (LIANG, 1999) and the authors further study, the authors sum up that the tetrahedral volume coordinates have the following characteristics:

(1) Anywhere of the moving point P inside or outside of the tetrahedron, u + v + w + r = 1.0; that is, u, v, w, r there are only three independent variables.

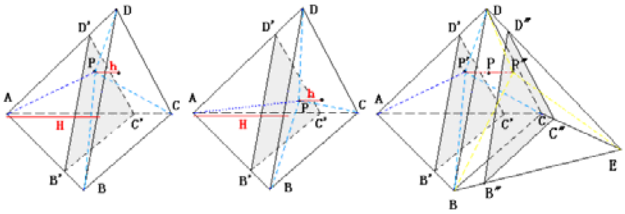
(2) To determine whether the moving point is inside or outside of the tetrahedron, a simple way is to check: | u | + | v | + | w | + | r | value, if the value equal to 1, the moving point is inside of the tetrahedron, other than outside of the Tetrahedron outside. And, according to the value of u, v, w, r, it could be determined that the position of P is in which region of tetrahedral.

(3) The moving point P at the vertexes A, B, C, D correspond to the coordinates (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1);

(4) The moving point P in the boundary surface BDC, ACD, ADB, ABC on the coordinates (0, v, w, r), (u, 0, w, r), (u, v, 0, r), (u, v, w, 0);

(5) The moving point P on the AB edge change from A to B, the value of the coordinates (u, v, 0,0) by (1,0,0,0) changed to (0,1,0,0) . The edges of AC, AD, BC, BD and CD have the similar characteristics.

(6) When the moving point P is away from a bottom surface, such as the BDC, with a distance h (i.e., moving point in the plane B'D'C' parallel to the bottom plane BDC), then u = h / H, v + w + r = 1-h / H value for (H to A distance from the bottom of the BDC),(Fig.12). The nature is very useful in ensuring partial derivative continuous in the two sides of interface. For example, tt can be used to ensure derivation function continuous in direction u on the interface of BDC, which is between two tetrahedrons ABCD and BECD (Fig.13).



(1) (2)

Figure 8. (1)Moving Points in plane B'C'D' have the same values of u. (2) Volume function derivation to u on the interface between two adjacent tetrahedrons

(7) Tetrahedral volume coordinates are natural coordinates, with the affine invariant, which coordinate value does not change with the coordinate system rotation (Rectangular coordinates do not having this nature), which is very beneficial to build a local coordinate system for tilted strata.
(8) The formula of calculation rectangle coordinate (x, y, z) from tetrahedral volume coordinate (u, v, w, r) is:
The formula calculation rectangle coordinate (x, y, z) from tetrahedral volume coordinate (u, v, w, r) is of conversion between the rectangular coordinates (x, y, z) and the tetrahedral volume coordinates (u, v, w, r) is:

$$\begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ X_A & X_B & X_C & X_D \\ Y_A & Y_B & Y_C & Y_D \\ Z_A & Z_B & Z_C & Z_D \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ r \end{pmatrix} \tag{21}$$

Or

$$\begin{aligned} &u + v + w + r = 1 \\ &x = X_A.u + X_B.v + X_C.w + X_D.r \\ &y = Y_A.u + Y_B.v + Y_C.w + Y_D.r \\ &z = Z_A.u + Z_B.v + Z_C.w + Z_D.r \end{aligned} \tag{22}$$

That is to say, the conversion between the rectangular coordinates (x, y, z) and the tetrahedral volume coordinates (u, v, w, r) is linear relationship. It is very useful to operation of derivation and integration.
To sum up, the tetrahedron volume coordinates is very similar to the triangle area, of the beautiful nature and easy transformation with rectangular coordinate. It should be a preferred coordinate system in expression and fitting volume function.

3.3 Hexahedron volume coordinate

This work comes from Yuqiu(1999, 2001, 2007) and Song (2007, 2008).

Numbering of vertex and face of a hexahedral element is shown in Figure 9.
Vertex coding and face encoding required to meet a strict rule that the four vertex formed a face nodes should be ordered according to right hand law (the thumb pointing to the outside normal direction), and to coding with the smallest numbered vertex as starting vertex. Therefore, the face 1 is constituted by 1584 vertexes; face 2 is constituted by 2376 vertexes; face 3 is constituted by 1265 vertexes; face 4 is constituted by 3487 vertexes; face 5 by is constituted by 5678 vertexes; face 6 is constituted by 1432 vertexes.

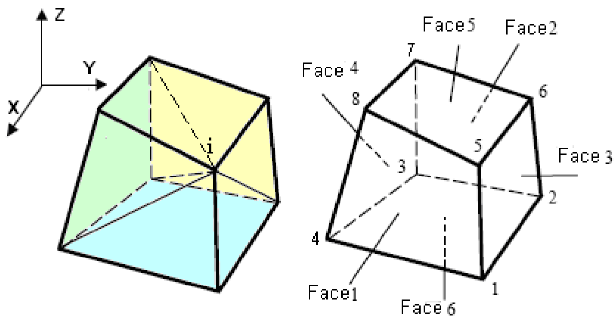


Figure 9. Numbering of vertex and face of a hexahedral element

The first category is similar to quadrilateral area coordinates, in order to describe the different shapes hexahedron must be the definition of the 24 non-dimensional shape parameter:

$$g_{iJ} = \frac{V_{iJ}}{V} \tag{23}$$

Of which:
i is vertex coding number (i=1~8);
J is face coding number, which the faces not passing through the vertex i (do not go through a vertex of the face there are three) encoding;
V_{iJ} is polyhedral volume composed by vertex i with face J.
V is hexahedron volume.

If the face I, face J, face K are the three faces that dose not pass the vertex i, the characteristic parameters which associated with the vertex i, can be written as g_{iI}, g_{iJ} and g_{iK}. They meet the following relationship:

$$g_{iI}, + g_{iJ} + g_{iK} = 1 \tag{24}$$

For different values of shape parameters, the hexahedral has different shape.

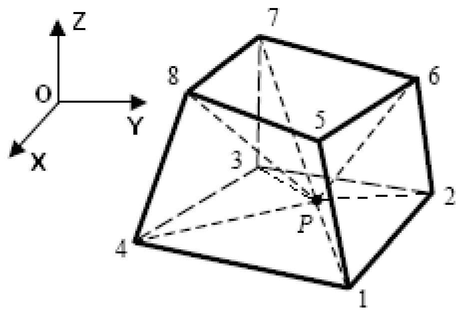


Figure 10. Moving point P in hexahedron

Figure 10 shows a any moving point P in the hexahedron. The hexahedron volume coordinates (L_1, L_2, L_3, L_4, L_5 and L_6) can be defined as:

$$L_I = \frac{V_I}{V} \quad \text{(I=1~6)} \tag{25}$$

Of which: V_1, V_2, V_3, V_4, V_5 and V_6 is pyramid volume enclosed by the moving point P and face 1, face 2, face 3, face 4, face 5 and face 6 respectively. Obviously there are:

$$L_1+L_2+L_3+L_4+L_5+L_6=1 \tag{26}$$

The volume coordinate of the vertexes are:

vertex1：(0, g_{12} ,0, g_{14}, g_{15}, 0),
vertex 2：(g_{21}, 0, 0, g_{24}, g_{25}, 0),
vertex 3：(g_{32}, 0, g_{33}, 0, g_{35}, 0);
vertex 4：(0, g_{42}, g_{43}, 0, g_{45}, 0);
vertex 5：(0, g_{52}, 0, g_{54}, 0, g_{56});
vertex 6：(g_{61}, 0, 0, g_{64}, 0, g_{66});
vertex 7：(g_{71}, 0, g_{73}, 0, 0, g_{76});
vertex 8：(0, g_{82}, g_{83}, 0, 0, g_{86});

It can prove that the definition of hexahedral volume coordinates with the rectangular coordinates have the following linear relationship:

$$L_I=a_Ix+b_Iy+c_Iz+d_I \qquad (I=1\sim6) \tag{27}$$

Here,

$$a_I=\frac{(y_k-y_i)(z_j-z_l)-(y_j-y_l)(z_k-z_i)}{6V}$$
$$b_I x\frac{(z_k-z_i)(x_j-x_l)-(z_j-z_l)(x_k-x_i)}{6V} \tag{28}$$
$$c_I=\frac{(x_k-x_i)(y_j-y_l)-(x_j-x_l)(y_k-y_i)}{6V}$$
$$d_I=[(x_i-x_k)(y_jz_i-y_iz_j)+(y_i-y_k)(z_jx_i-z_ix_j)+(z_i-z_k)(x_jy_i-x_iy_j)]/(6V)$$

Where, i, j, k and l is the vertex which constituted the face I (in the order of i, j, k and l).

4. USAGE OF AREA/VOLUME COORDINATE IN GIS

4.1 Usage of Area Coordinate

The area coordinate (including the triangle area coordinate and the quadrilateral area coordinate) may by used in true 3D GIS to express the nature surface which may be perpendicular. The natures of triangle or quadrilateral area coordinate are excellent in deal with smooth connection problems of high-order (quadric or cubic).surface.

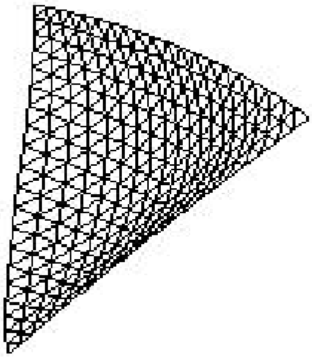


Figure 11. Triangle area coordinate express nature surface

4.2 Usage of volume Coordinate

The volume coordinate may be used in volume function model to express property ununiformity inside of solid element, as shown in figure 11.

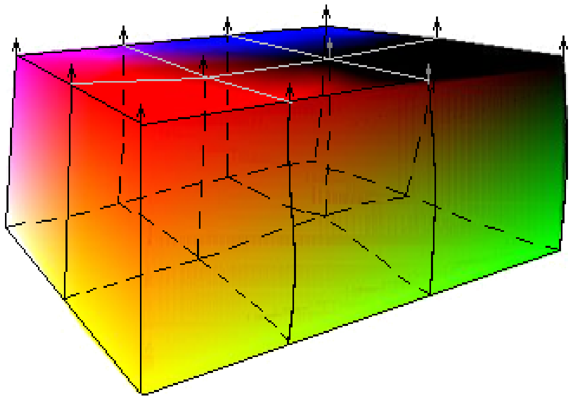


Figure 12. Volume coordinate may be used in function model to present ununiformity peroperty inside of solid feature

5. CONCLUSION

The paper introduces research work about area coordinate and volume coordinate theory. The area/volume coordinate theories have been used in limited element method of engineering mechanics and computer geometry modelling. The theories have a great application potential in 3D GIS. But the work in GIS science is just in initial stages. The authors have another paper "Volume Function Data Model Based on Volume Coordinate for True 3D GIS" to discuss the theory about the volume function model.

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