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GÖTTERDÄMMERUNG OVER LEAST SQUARES ADJUSTMENT

TORBEN KRARUP, THE GEODETIC INSTITUTE, COPENHAGEN, DENMARK
JENS JUHL AND KURT KUBIK, AALBORG UNIVERSITY CENTRE,
AALBORG/DENMARK

Abstract:

With the inclusion of systematic image errors and automatic techniques of gross error detection the least squares photogrammetric adjustment has reached an end stage in development, in which an objective determination and separation of "systematic" and "gross" errors becomes impossible.

The authors propose new lines of thought which resolve this deadlock and allow the direct allocation of various error sources.

GÖTTERDÄMMERUNG

OVER

LEAST SQUARES ADJUSTMENT

1. Introduction

The method of least squares became the generally accepted computation method in geodesy and photogrammetry. However, it has serious drawbacks. It is particularly ineffective in the detection and location of gross errors and systematic errors in the measurements. Although an advanced test theory was developed for this purpose, an economic semiautomatic detection of these errors could not be achieved.

However, other methods exist which give much more reliable results in the presence of outlying measurements. These methods are based on minimizing other objective functions and are discussed below.

2. Drawbacks of the Method of Least Squares

As generally known, the method of least squares minimizes the sum of squares of corrections v to our measurements

$$\sum v^2 \rightarrow \min. \qquad (1)$$

From the adjustment results and the residuals (-v) it can be extraordinarily difficult to detect and locate gross errors. In fact, the least squares method is very efficient in hiding large errors and in distributing their effects over many measurements, thus making them unrecognizable. Erroneous measurements are not necessarily those which have the largest residuals after adjustment. This property of the least squares method is demonstrated in appendix A for an example of numerical relative orientation, where a gross error of 40 μm is reduced to the largest residual of 7 μm in another measurement.

In order to cope with these problems, an advanced test theory was developed by Baarda, based on the ideas of Scheffé. This test method is, however, very elaborate, requiring either the inversion of the normal equation matrix or repeated adjustments with successive exclusion of all the individual measurements. Moreover - in the presence of more than one gross error - the method does not ensure correct results, because the decision of excluding an "erroneous" measurement is not reviewed during the further computation.

Unknown systematic errors in measurements like lens- and film distortion in photogrammetry create another problem in least squares. When applying the least squares method to measurements with systematic errors, the result is nearly randomly distributed residuals. This effect is, for example, well known in photogrammetric block adjustment. In practical computations different alternative hypotheses (extra parameters) regarding the systematic errors are included in adjustment and from the results it is judged which hypothesis is most probable. This evaluation is difficult due to the strong mutual correlation between the estimated parameters for systematic errors and their strong correlation to the other unknowns in the adjustment. Moreover, completeness is not guaranteed: if a certain type of systematic error is not included in the adjustment, this error will not be found.

3. Least Sum Method

As an alternative to the least squares method, the least sum method was proposed already in 1887 (Edgeworth, 1887). Here, the sum of the corrections v is minimized

$$\sum |v| \rightarrow \min \qquad (2)$$

$|\cdot|$ absolute value of $\cdot$

The method never found larger application due to difficulties in its numerical computation: a solution of a linear programming problem is required. In recent years, however, with the introduction of efficient algorithms (simplex algorithm), the solution of the least sum principle does not require more time than least squares.

This method is intuitively attractive, because larger residuals are easier tolerated, thus facilitating detection and location of outlying measurements from the results. In papers of Barrodale (1968), who strongly advocated the method in recent years, the method gives consistently better results in the presence of outliers than least squares. In most cases the individual residuals clearly indicate which measurements are erroneous. Only in the presence of many and unfavourably located gross errors the method may lead to wrong conclusions. The power of the method is demonstrated in our small example in appendix A, where the introduced gross error is now clearly visible in the residuals. The method, however, still lacks adequate theoretical foundation. For an easy numerical solution of this principle, refer to appendix B.

4. Robust Estimators

Another alternative to least squares are robust estimators, introduced by Kendall in 1948. Robust estimators are estimators which are relatively insensitive to limited variations in the distribution function of the measurements, and thus to the presence of gross and systematic errors.

There exists a large class of robust estimation principles (about seventy). The most well known among those are proposed by Huber and Hampel, and consist of minimizing

$$\varphi(v) \rightarrow \min, \qquad (3)$$

with $\varphi(v)$ being for Huber

$$\varphi(v) = \begin{cases} v^2 \text{ if } |v| \leq 2\sigma \\ 2\sigma\ (2|v|-2\sigma) \text{ if } |v| > 2\sigma \end{cases} \qquad (3A)$$

σ being standard deviation of measurements,

and for Hampel

$$\frac{\partial\varphi}{\partial v} = \text{sign}(v) * \begin{cases} |v| & 0 \leq |v| < a \\ a & a \leq |v| < b \\ \frac{c-|v|}{c-b} \cdot a & b \leq |v| < c \\ 0 & |v| \geq c \end{cases} \qquad (3B)$$

a,b,c, being constants.

Note that in both cases the adjustment principle depends on the magnitude of the correction v, with larger corrections contributing only little to the objective function.

Another robust estimation principle is given by

$$\sum |v|^p \rightarrow \min \qquad 1 \leq p < 2 \qquad (4)$$

where the most favourable range of values p is between 1.2 and 1.5.

The concept of robust estimation is so new that no united theory exists which enables us to select the best adjustment principle for our particular geodetic problems. But experimentally, it was found that these methods are by far superior to least squares in the detection and location of gross and systematic errors. They also compare favourably to the least sum method. As an illustration of the robust estimation principles, appendix A includes the computation of the relative orientation according to principle (4). The results are in this particular case slightly inferior to the least sum method.

In practical computations it is advocated to use both the least squares method and one of the alternative principles. A larger difference in the results indicates the presence of gross errors or systematic errors and requires more detailed analysis of the measurements.

For a simple computational algorithm for robust estimation, refer to appendix B.

5. The Danish Method - An Extension of the Robust Estimation Principle

The above difficulties with the solution of the least squares method have since many years been recognized by the Geodetic Institute of Denmark, where since early seventies an automatic error search routine has been used in the computation of all larger geodetic problems. This method was developed after the ideas of Krarup (1967) and is especially designed to eliminate gross errors. The starting point of the method is a conventional least squares adjustment. From the residuals of this first adjustment, new weights are computed for the individual measurements, based on the weight function

$$p = \begin{cases} 1 & \text{for } |v| \leq 2\sigma \\ \text{proportional to } \exp(-c.v^2) & \text{for } |v| > 2\sigma \end{cases}$$

σ being standard deviation of measurements
c being constant.

With these weights, a new least squares adjustment is computed, and this process of reweighting and adjustment is repeated until convergence is achieved. This is usually the case after 5-10 iterations. Finally, measurements affected by gross errors have weights 0 and their residuals are a measure for the magnitude of the errors.

This method proved to be extremely effective in dealing with erroneous data. Simulation-runs indicate that this method, with properly chosen weight sequences, is more effective than the other alternatives to least squares. When applying this method to our small example in appendix A, the exact amount of the gross error is recovered after only 6 iterations.

The method may be interpreted as an iterative solution to the Bayesian estimation principle

$$\text{Var}(v) = \sum v^2 \exp(-\frac{v^2}{2\sigma^2}) \rightarrow \min$$

minimizing the variance of uncorrelated, normally distributed measurements.

Another possible interpretation of the Danish method is given by nonlinear programming: Find the largest number of measurements, which is mutually consistent, and use only these measurements in the least squares adjustment to determine the unknowns x. An alternative (dual) formulation is to find those observations which are not consistent with the majority and exclude them from adjustment. In formal notation this problem writes as

$$\max_k [v_k] \qquad [\cdot] \text{ number of } \cdot \quad k \in \{1 \ldots n\}$$

$$|v_k| < 2\sigma$$

$$\sum v_k^2(x) = \min_\xi (\sum v_k^2(\xi)).$$

The numerical solution is complicated by the interrelationship of the two optimization problems. The iterative method of the Geodetic Institute may be interpreted as a penalty method for solution of the above problem, and an optimal weight sequence assuring rapid convergence can be derived. A possible weight selection, usually yielding good results in photogrammetry, is shown below:

1st iteration: $p = 1$

2nd and 3rd iteration: $p = (\exp[-(\frac{v}{\sigma})^{4.4}])^{0.05}$

following iterations: $p = (\exp[-(\frac{v}{\sigma})^{3.0}])^{0.05}$.

6. Stein Estimator - A More Accurate Estimator than Least Squares

James and Stein (1961) proved in a classical paper that there exist more accurate estimators than least squares, even for the case of normal distribution and in the absence of gross errors. When knowing a priori the accuracy of the measurements, an estimator $\tilde{v}$ for the corrections may be constructed from the least squares estimator v by the rule

$$\tilde{v} = (I-A)v$$

where I denotes the unit matrix and A is a positive definite matrix depending on σ and v. This estimator has a smaller error variance than v. The above-mentioned alternative methods can partly be classified under this principle, thus yielding more accurate results than least squares, even in the absence of gross and systematic errors.

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Appendix A: Results of Different Adjustment Principles

Consider numerical relative orientation of a pair of photographs, based on the wellknown projective relationships.

The lefthand camerastation is regarded to be fixed. Below the measured image coordinates x and y for 16 points are presented, used in relative orientation, and the results of the individual adjustments. Note that the first adjustment was executed without gross error, and that a gross error of 40 μm was introduced in the subsequent adjustments at point 100. The least squares adjustment completely camouflages this error. The least sum method and robust estimation retrieve 3/4 of the error, the Danish method also finds its correct amount.

Adjustment according to least squares method

PUNKTNR	BILLEDNR	X' (MM)	Y' (MM)	VX' (UM)	VY' (UM)
100	1	-100.0000	100.0000	0.0	0.2
	2	0.0000	100.0000	-0.0	-0.2
101	1	0.0000	100.0000	0.0	0.6
	2	100.0000	99.9960	-0.0	-0.6
102	1	0.0000	60.0000	0.0	-1.1
	2	100.0000	60.0000	0.0	1.1
103	1	-100.0000	40.0000	0.0	-2.1
	2	0.0000	40.0040	-0.0	2.1
104	1	C.0000	40.0000	0.0	1.1
	2	100.0000	39.9960	-0.0	-1.1
105	1	-100.0000	20.0000	0.0	1.3
	2	0.0000	19.9970	-0.0	-1.3
106	1	0.0000	20.0000	0.0	0.3
	2	100.0000	19.9980	-0.0	-0.3
107	1	-100.0000	0.0000	-0.0	-0.3
	2	0.0000	0.0001	0.0	0.3
108	1	0.0000	0.0001	0.0	-1.0
	2	100.0000	0.0010	-0.0	1.0
109	1	-100.0000	-40.0000	0.0	2.1
	2	0.0000	-40.0050	-0.0	-2.1
110	1	0.0000	-40.0000	0.0	0.9
	2	100.0000	-40.0020	-0.0	-0.9
111	1	-100.0000	-60.0000	0.0	1.0
	2	0.0000	-60.0030	-0.0	-1.0
112	1	0.0000	-60.0000	0.0	-2.4
	2	100.0000	-59.9950	-0.0	2.4
113	1	-100.0000	-80.0000	0.0	-1.6
	2	0.0000	-79.9980	-0.0	1.6
114	1	0.0000	-80.0000	0.0	0.7
	2	100.0000	-80.0010	-0.0	-0.7
115	1	-100.0000	-100.0000	-0.0	-0.6
	2	0.0000	-100.0000	-0.0	0.6
116	1	0.0000	-100.0000	0.0	0.9
	2	100.0000	-100.0010	-0.0	-0.9

Numerical relative orientation without gross error

Adjustment according to least squares method

POINT NO	PHOTO NO	X' (MM)	Y' (MM)	VX' (UM)	VY' (UM)
100	2	0.0000	99.9600	.0	-5.6
	1	-100.0000	100.0000	.0	5.6
101	2	100.0000	99.9960	-.0	3.2
	1	0.0000	100.0000	.0	-3.2
102	2	100.0000	60.0000	-.0	1.0
	1	0.0000	60.0000	.0	-1.0
103	2	0.0000	40.0040	-.0	7.3
	1	-100.0000	40.0000	.0	-7.3
104	2	100.0000	39.9960	.0	-2.4
	1	0.0000	40.0000	-.0	2.4
105	2	0.0000	19.9970	-.0	1.7
	1	-100.0000	20.0000	.0	-1.7
106	2	100.0000	19.9980	.0	-2.3
	1	0.0000	20.0000	-.0	2.3
107	2	0.0000	0.0000	-.0	1.5
	1	-100.0000	0.0000	.0	-1.5
108	2	100.0000	.0010	.0	-1.3
	1	0.0000	0.0000	-.0	1.3
109	2	0.0000	-40.0050	.0	-3.0
	1	-100.0000	-40.0000	-.0	3.0
110	2	100.0000	-40.0020	.0	-2.4
	1	0.0000	-40.0000	-.0	2.4
111	2	0.0000	-60.0030	.0	-2.3
	1	-100.0000	-60.0000	-.0	2.3
112	2	100.0000	-59.9950	-.0	1.9
	1	0.0000	-60.0000	.0	-1.9
113	2	0.0000	-79.9980	-.0	.4
	1	-100.0000	-80.0000	.0	-.4
114	2	100.0000	-80.0010	-.0	.2
	1	0.0000	-80.0000	.0	-.2
115	2	0.0000	-100.0000	.0	-.1
	1	-100.0000	-100.0000	-.0	.1
116	2	100.0000	-100.0010	-.0	2.0
	1	0.0000	-100.0000	.0	-2.0

Numerical relative orientation with gross error of 40 µm point 100, photo No 2.

Adjustment according to least sum

PUNKTNR	BILLEDNR	X' (MM)	Y' (MM)	VX' (UM)	VY' (UM)
100	1	-100.0000	100.0000	0.0	13.9
	2	0.0000	99.9600	-0.0	-13.9
101	1	0.0000	100.0000	0.0	-0.1
	2	100.0000	99.9960	-0.0	0.1
102	1	0.0000	60.0000	0.0	-0.6
	2	100.0000	60.0000	-0.0	0.6
103	1	-100.0000	40.0000	0.0	-4.5
	2	0.0000	40.0040	-0.0	4.5
104	1	0.0000	40.0000	-0.0	1.9
	2	100.0000	39.9960	0.0	-1.9
105	1	-100.0000	20.0000	0.0	-0.2
	2	0.0000	19.9970	-0.0	0.2
106	1	0.0000	20.0000	-0.0	1.2
	2	100.0000	19.9980	0.0	-1.3
107	1	-100.0000	0.0000	0.0	-1.0
	2	0.0000	0.0001	-0.0	1.0
108	1	0.0000	0.0001	0.0	-0.0
	2	100.0000	0.0010	-0.0	0.0
109	1	-100.0000	-40.0000	-0.0	2.4
	2	0.0000	-40.0050	0.0	-2.4
110	1	0.0000	-40.0000	-0.0	1.3
	2	100.0000	-40.0020	0.0	-1.3
111	1	-100.0000	-60.0000	-0.0	1.6
	2	0.0000	-60.0030	0.0	-1.6
112	1	0.0000	-60.0000	0.0	-2.5
	2	100.0000	-59.9950	-0.0	2.5
113	1	-100.0000	-80.0000	0.0	-0.9
	2	0.0000	-79.9980	-0.0	0.9
114	1	0.0000	-80.0000	0.0	0.0
	2	100.0000	-80.0010	0.0	-0.0
115	1	-100.0000	-100.0000	0.0	0.0
	2	0.0000	-100.0000	0.0	-0.0
116	1	0.0000	-100.0000	0.0	-0.6
	2	100.0000	-100.0010	-0.0	0.6

Numerical relative orientation with gross error of 40 µm point 100, photo No 2.

Adjustment according to robust estimation

PUNKTNR	BILLEDNR	X' (MM)	Y' (MM)	VX' (UM)	VY' (UM)
100	1	-100.0000	100.0000	0.0	11.1
	2	0.0000	99.9600	-0.0	-11.1
101	1	0.0000	100.0000	0.0	-1.1
	2	100.0000	99.9960	-0.0	1.1
102	1	0.0000	60.0000	0.0	-1.0
	2	100.0000	60.0000	-0.0	1.0
103	1	-100.0000	40.0000	0.0	-5.9
	2	0.0000	40.0040	-0.0	5.9
104	1	0.0000	40.0000	-0.0	1.7
	2	100.0000	39.9960	0.0	-1.7
105	1	-100.0000	20.0000	0.0	-1.2
	2	0.0000	19.9970	-0.0	1.2
106	1	0.0000	20.0000	-0.0	1.2
	2	100.0000	19.9980	0.0	-1.2
107	1	-100.0000	0.0000	0.0	-1.7
	2	0.0000	0.0001	-0.0	1.7
108	1	0.0000	0.0001	-0.0	0.1
	2	100.0000	0.0010	0.0	-0.1
109	1	-100.0000	-40.0000	-0.0	2.2
	2	0.0000	-40.0050	0.0	-2.2
110	1	0.0000	-40.0000	-0.0	1.5
	2	100.0000	-40.0020	0.0	-1.5
111	1	-100.0000	-60.0000	-0.0	1.5
	2	0.0000	-60.0030	0.0	-1.6
112	1	0.0000	-60.0000	0.0	-2.4
	2	100.0000	-59.9950	-0.0	2.4
113	1	-100.0000	-80.0000	0.0	-0.8
	2	0.0000	-79.9980	-0.0	0.8
114	1	0.0000	-80.0000	0.0	0.0
	2	100.0000	-80.0010	0.0	-0.0
115	1	-100.0000	-100.0000	-0.0	0.1
	2	0.0000	-100.0000	0.0	-0.1
116	1	0.0000	-100.0000	0.0	-0.8
	2	100.0000	-100.0010	-0.0	0.8

Numerical relative orientation with gross error of 40 μm point 100, photo No 2.

Adjustment according to the Danish method

POINT NO	PHOTO NO	X' (MM)	Y' (MM)	VX' (UM)	VY' (UM)	WEIGHT
100	2	0.0000	99.9600	-.0	-20.5	.0
	1	-100.0000	100.0000	.0	20.5	.0
101	2	100.0000	99.9960	-.0	-.7	
	1	0.0000	100.0000	.0	.7	
102	2	100.0000	60.0000	.0	1.1	
	1	0.0000	60.0000	.0	-1.1	
103	2	0.0000	40.0040	-.0	2.0	
	1	-100.0000	40.0000	.0	-2.0	
104	2	100.0000	39.9960	-.0	-1.1	
	1	0.0000	40.0000	.0	1.1	
105	2	0.0000	19.9970	-.0	-1.4	
	1	-100.0000	20.0000	.0	1.4	
106	2	100.0000	19.9980	-.0	-.2	
	1	0.0000	20.0000	.0	.2	
107	2	0.0000	0.0000	.0	.2	
	1	-100.0000	0.0000	-.0	-.2	
108	2	100.0000	.0010	-.0	1.1	
	1	0.0000	0.0000	.0	-1.1	
109	2	0.0000	-40.0050	-.0	-2.1	
	1	-100.0000	-40.0000	.0	2.1	
110	2	100.0000	-40.0020	-.0	-.8	
	1	0.0000	-40.0000	.0	.8	
111	2	0.0000	-60.0030	-.0	-1.0	
	1	-100.0000	-60.0000	.0	1.0	
112	2	100.0000	-59.9950	-.0	2.5	
	1	0.0000	-60.0000	.0	-2.5	
113	2	0.0000	-79.9980	-.0	1.6	
	1	-100.0000	-80.0000	.0	-1.6	
114	2	100.0000	-80.0010	-.0	-.8	
	1	0.0000	-80.0000	.0	.8	
115	2	0.0000	-100.0000	-.0	.6	
	1	-100.0000	-100.0000	-.0	-.6	
116	2	100.0000	-100.0010	-.0	-1.0	
	1	0.0000	-100.0000	.0	1.0	

Numerical relative orientation with gross error of 40 μm point 100, photo No 2.

Appendix B: Computational Procedure for Alternative Adjustment

A simple computational routine for the alternative adjustment methods, which is closely related to the least squares method, is given in the following:

For solving the adjustment problems (3) and (4),

$$\varphi(v) \rightarrow \min$$

solves repeatedly the weighted least squares problem with weights for the individual measurements equal to unity in the first iteration and equal to

$$p = \frac{\varphi(v)}{v^2 + c}$$ c constant, relatively small as compared to v^2

in the subsequent iterations. In particular, for the least sum method, $p = 1/(|v| + c)$. Convergence can be proved under mild conditions for $c \rightarrow 0$.